On approximation to a real number by algebraic numbers of bounded degree

Anthony Poëls

Diophantine approximation and transcendence
CIRM, Luminy

12th November 2025

Institut Camille Jordan
Université Claude Bernard Lyon 1

Introduction

How can a real number be "well" approximated by rational numbers?

Introduction

How can a real number be "well" approximated by rational numbers?

Fix $\xi \in \mathbb{R} \setminus \mathbb{Q}$. There are infinitely many $p/q \in \mathbb{Q}$ such that

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

Introduction

How can a real number be "well" approximated by rational numbers?

Fix $\xi \in \mathbb{R} \setminus \mathbb{Q}$. There are infinitely many $p/q \in \mathbb{Q}$ such that

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

Remark: q measure the "complexity" (= the "height") of the rational number p/q.

Introduction

Generalization? Rational numbers = algebraic numbers of degree 1.

Introduction

Generalization? Rational numbers = algebraic numbers of degree 1.

Question. Fix an integer $n \geq 1$. Are there ∞-many (real or complex) algebraic numbers α, of degree at most n, such that

$$|\xi - \alpha| \overset{??}{\ll} \frac{1}{H(\alpha)^{n+1}},$$

Introduction

Generalization? Rational numbers = algebraic numbers of degree 1.

Question. Fix an integer $n \geq 1$. Are there ∞-many (real or complex) algebraic numbers α, of degree at most n, such that

$$|\xi - \alpha| \overset{??}{\ll} \frac{1}{H(\alpha)^{n+1}},$$

(here $H(\alpha) \overset{\mathrm{def}}{=} \|P_\alpha\| =$ *naive height of* α, where $P_\alpha \in \mathbb{Z}[X] =$ minimal polynomial of α, irr. over $\mathbb{Z}$).

Introduction

Generalization? Rational numbers = algebraic numbers of degree 1.

Question. Fix an integer $n \geq 1$. Are there ∞-many (real or complex) algebraic numbers α, of degree at most n, such that

$$|\xi - \alpha| \overset{??}{\ll} \frac{1}{H(\alpha)^{n+1}},$$

(here $H(\alpha) \overset{\text{def}}{=} \|P_\alpha\| =$ *naive height of* α, where $P_\alpha \in \mathbb{Z}[X]$ = minimal polynomial of α, irr. over $\mathbb{Z}$).

Example: If p/q irreducible, then $H(p/q) = \|(p,q)\| \asymp |q|$.

Diophantine exponents

A diophantine exponent

Define $\omega_n^*(\xi) =$ as the sup. of all $\omega \geq 0$ such that there are ∞–many algebraic numbers α, of degree at most n, satisfying

$$|\xi - \alpha| \leq \frac{1}{H(\alpha)^{\omega+1}}.$$

Diophantine exponents

A diophantine exponent

Define $\omega_n^*(\xi) =$ as the sup. of all $\omega \geq 0$ such that there are ∞–many algebraic numbers α, of degree at most n, satisfying

$$|\xi - \alpha| \leq \frac{1}{H(\alpha)^{\omega+1}}.$$

Some properties of ω_n^*.

- $\omega_n^*(\xi) = n$ for almost every $\xi \in \mathbb{R}$ (Sprindžuk, 1969).
- If ξ is algebraic of degree d, then $\omega_n^*(\xi) = \min\{n, d-1\}$ (Schmidt's subspace theorem).
- $[n, \infty] \subseteq \omega_n^*(\mathbb{R} \setminus \overline{\mathbb{Q}})$ (Baker-Schmidt, 1970).

Wirsing's problem (1961)

Theorem (Wirsing, 1961)

For any transcendental real number ξ, we have

$$\omega_n^*(\xi) \geq \frac{n+1}{2} \tag{4}$$

and

$$\omega_n^*(\xi) \geq W(n) = \frac{n}{2} + 1 - o(1). \tag{9}$$

Wirsing's problem (1961)

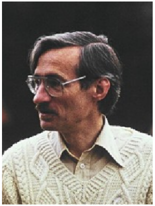
Theorem (Wirsing, 1961)

For any transcendental real number ξ, we have

$$\omega_n^*(\xi) \geq \frac{n+1}{2} \tag{4}$$

and

$$\omega_n^*(\xi) \geq W(n) = \frac{n}{2} + 1 - o(1). \tag{9}$$



Auch (9) ließe sich noch verbessern. Da die Verbesserung aber geringfügig ist und kein Grund besteht, anzunehmen, daß man mit (4) und (9) schon in der Nähe der bestmöglichen Schranke ist, soll darauf nicht eingegangen werden. Vielleicht gilt sogar stets $w_n^*(\xi) \geqq n$ für $n < s$ und reelles ξ.

E. Wirsing, 1961, Approximation mit algebraischen Zahlen beschränkten Grades, p.69.

Wirsing's problem (1961)

"Wirsing's conjecture". We have $\omega_n^*(\xi) \geq n$ for any $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$.

Wirsing's problem (1961)

"Wirsing's conjecture". We have $\omega_n^*(\xi) \geq n$ for any $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$.

- $n = 1$: true (approximation by rational numbers).
- $n = 2$ (Davenport and Schmidt, 1967) : true.
- $n \geq 3$: (widely) open problem.

Wirsing's problem (1961)

"Wirsing's conjecture". We have $\omega_n^*(\xi) \geq n$ for any $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$.

- $n = 1$: true (approximation by rational numbers).
- $n = 2$ (Davenport and Schmidt, 1967) : true.
- $n \geq 3$: (widely) open problem.

Brief history

$$\omega_n^*(\xi) \geq \begin{cases} 0.5n + 1 - o(1) & \text{Wirsing, 1961} \\ n/2 + 3 - o(1) & \text{Bernik and Tishchenko, } < 2021 \\ n/\sqrt{3} \approx 0.577n & \text{Badziahin-Schleischitz, 2021.} \end{cases}$$

Wirsing's problem (1961)

"Wirsing's conjecture". We have $\omega_n^*(\xi) \geq n$ for any $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$.

- $n = 1$: true (approximation by rational numbers).
- $n = 2$ (Davenport and Schmidt, 1967) : true.
- $n \geq 3$: (widely) open problem.

Brief history

$$\omega_n^*(\xi) \geq \begin{cases} 0.5n + 1 - o(1) & \text{Wirsing, 1961} \\ n/2 + 3 - o(1) & \text{Bernik and Tishchenko, } < 2021 \\ n/\sqrt{3} \approx 0.577n & \text{Badziahin-Schleischitz, 2021.} \end{cases}$$

Main Theorem (P., 2025)

For any $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$, we have

$$\omega_n^*(\xi) \geq \frac{n}{2 - \log 2} \approx 0.765n.$$

Main obstruction

Dirichlet's Theorem (1842)

Let $n \geq 1$ be an integer and let $\xi \in \mathbb{R}$. For each $X > 1$ there is $(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$ such that

$$|\underbrace{a_0 + a_1\xi + \cdots + a_n\xi^n}_{P(\xi)}| \leq X^{-n} \quad \text{and} \quad \underbrace{\max_{1\leq i\leq n} |a_i|}_{\asymp \|P\|} \leq X.$$

Main obstruction

Dirichlet's Theorem (1842)

Let $n \geq 1$ be an integer and let $\xi \in \mathbb{R}$. For each $X > 1$ there is $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$ such that

$$|\underbrace{a_0 + a_1\xi + \cdots + a_n\xi^n}_{P(\xi)}| \leq X^{-n} \quad \text{and} \quad \underbrace{\max_{1 \leq i \leq n} |a_i|}_{\asymp \|P\|} \leq X.$$

Naive approach.

- Construct ∞–many P in $\mathbb{Z}[X]_{\leq n}$ with $|P(\xi)| \ll \|P\|^{-n}$ (Dirichlet's Theorem).
- Consider α root of P such that $|\xi - \alpha|$ is minimal (we have $H(\alpha) \ll \|P\|$).
- **Remark:** If $P(\xi)$ is "small", then so is $|\xi - \alpha|$ (and vice versa).

Main obstruction

Naive approach.

- Estimate of $|\xi - \alpha|$?

Main obstruction

Naive approach.

- Estimate of $|\xi - \alpha|$? We have

$$|\xi - \alpha| \leq n\frac{|P(\xi)|}{|P'(\xi)|} \ll \frac{H(\alpha)^{-n}}{|P'(\xi)|}.$$

Main obstruction

Naive approach.

- Estimate of $|\xi - \alpha|$? We have

$$|\xi - \alpha| \leq n\frac{|P(\xi)|}{|P'(\xi)|} \ll \frac{H(\alpha)^{-n}}{|P'(\xi)|}.$$

Serious issue: $|P'(\xi)|$ could be "small" ! ($\Leftrightarrow$ P has at least two roots "close" to ξ).

Main obstruction

Naive approach.

- Estimate of $|\xi - \alpha|$? We have

$$|\xi - \alpha| \leq n\frac{|P(\xi)|}{|P'(\xi)|} \ll \frac{H(\alpha)^{-n}}{|P'(\xi)|}.$$

Serious issue: $|P'(\xi)|$ could be "small" ! ($\Leftrightarrow$ P has at least two roots "close" to ξ).

Principle : If $P(\xi)$ is "very small" and $|P'(\xi)|$ is "not too small", then α is a "rather good" algebraic approximation.

Wirsing's proof

Wirsing (1961):

$$\omega_n^*(\xi) \geq \frac{n+1}{2}.$$

Idea : if one good approximation is not enough, use two !

Wirsing's proof

Wirsing (1961):

$$\omega_n^*(\xi) \geq \frac{n+1}{2}.$$

Idea : if one good approximation is not enough, use two !

- Construction of P and Q : Dirichlet's theorem.

Wirsing's proof

Wirsing (1961):

$$\omega_n^*(\xi) \geq \frac{n+1}{2}.$$

Idea : if one good approximation is not enough, use two !

- Construction of P and Q : Dirichlet's theorem. We ask that

$$|P(\xi)| \ll \|P\|^{-n}, \quad |Q(\xi)| \ll \|P\|^{-n} \quad \text{and} \quad \|Q\| \ll \|P\|.$$

Wirsing's proof

Wirsing (1961):

$$\omega_n^*(\xi) \geq \frac{n+1}{2}.$$

Idea : if one good approximation is not enough, use two !

- Construction of P and Q : Dirichlet's theorem. We ask that

$$|P(\xi)| \ll \|P\|^{-n}, \quad |Q(\xi)| \ll \|P\|^{-n} \quad \text{and} \quad \|Q\| \ll \|P\|.$$

- Make sure that P and Q are coprime... (nice trick)

Wirsing's proof

Wirsing (1961):

$$\omega_n^*(\xi) \geq \frac{n+1}{2}.$$

Idea : if one good approximation is not enough, use two !

- Construction of P and Q : Dirichlet's theorem. We ask that

$$|P(\xi)| \ll \|P\|^{-n}, \quad |Q(\xi)| \ll \|P\|^{-n} \quad \text{and} \quad \|Q\| \ll \|P\|.$$

- Make sure that P and Q are coprime... (nice trick)
- Bound from above their resultant

$$1 \leq |\underbrace{\mathrm{Res}(P, Q)}_{\in \mathbb{Z} \setminus \{0\}}| = |\underbrace{\mathrm{Res}(P(X+\xi), Q(X+\xi))}_{\text{involves } P(\xi),\, P'(\xi),\, Q(\xi),\, Q'(\xi)}|.$$

Wirsing's proof

Wirsing (1961):

$$\omega_n^*(\xi) \geq \frac{n+1}{2}.$$

Idea : if one good approximation is not enough, use two !

- Construction of P and Q : Dirichlet's theorem. We ask that

$$|P(\xi)| \ll \|P\|^{-n}, \quad |Q(\xi)| \ll \|P\|^{-n} \quad \text{and} \quad \|Q\| \ll \|P\|.$$

- Make sure that P and Q are coprime... (nice trick)
- Bound from above their resultant

$$1 \leq |\underbrace{\mathrm{Res}(P,Q)}_{\in\mathbb{Z}\setminus\{0\}}| = |\underbrace{\mathrm{Res}(P(X+\xi),Q(X+\xi))}_{\text{involves } P(\xi),\, P'(\xi),\, Q(\xi),\, Q'(\xi)}|.$$

Write

$$P(X+\xi) = P(\xi) + P'(\xi)X + \frac{P''(\xi)}{2}X^2 + \cdots$$

Wirsing's proof

... then $\mathrm{Res}(P(X+\xi), Q(X+\xi))$ is equal to

$$\underbrace{\left|\begin{array}{cccc} P(\xi) & 0 & 0 & \dots \\ P'(\xi) & P(\xi) & 0 & \\ P''(\xi)/2 & P'(\xi) & P(\xi) & \\ \hline & & & \\ & \mathcal{O}(\|P\|) & & \\ & & & \end{array}\right.}_{\deg Q} \underbrace{\left.\begin{array}{cccc} Q(\xi) & 0 & 0 & \dots \\ Q'(\xi) & Q(\xi) & 0 & \\ Q''(\xi)/2 & Q'(\xi) & Q(\xi) & \\ \hline & & & \\ & \mathcal{O}(\|Q\|) & & \\ & & & \end{array}\right|}_{\deg P} .$$

Wirsing's proof

Putting everything together, we finally find that

$$1 \ll \max\{|P'(\xi)|, |Q'(\xi)|\}^2 \underbrace{\max\{|P(\xi)|, |Q(\xi)|\}}_{\ll H(P)^{-n}} \|P\|^{2n-3}.$$

Wirsing's proof

Putting everything together, we finally find that

$$1 \ll \max\{|P'(\xi)|, |Q'(\xi)|\}^2 \underbrace{\max\{|P(\xi)|, |Q(\xi)|\}}_{\ll H(P)^{-n}} \|P\|^{2n-3}.$$

- $\Rightarrow$ either $|P'(\xi)|$ or $|Q'(\xi)|$ is “not too small”;
- $\Rightarrow$ either P or Q has a root “quite close to” ξ ;

Wirsing's proof

Putting everything together, we finally find that

$$1 \ll \max\{|P'(\xi)|, |Q'(\xi)|\}^2 \underbrace{\max\{|P(\xi)|, |Q(\xi)|\}}_{\ll H(P)^{-n}} \|P\|^{2n-3}.$$

- $\Rightarrow$ either $|P'(\xi)|$ or $|Q'(\xi)|$ is "not too small";
- $\Rightarrow$ either P or Q has a root "quite close to" ξ ;

After computation:

$$\omega_n^*(\xi) \geq \frac{n+1}{2}.$$

To go further...

Our goal: prove that $\omega_n^*(\xi) \geq \dfrac{n}{2-\log 2} = 0.765\cdots n$.

Our strategy: "THE MORE, THE MERRIER." (14th-century, poem *Pearl*, line 850).

To go further...

Our goal: prove that $\omega_n^*(\xi) \geq \dfrac{n}{2-\log 2} = 0.765\cdots n$.

Our strategy: "THE MORE, THE MERRIER." (14th-century, poem *Pearl*, line 850).

$\rightarrow$ Choose $n+1$ linearly independent "good" polynomial approximations in $\mathbb{Z}[X]_{\leq n}$.

To go further...

Our goal: prove that $\omega_n^*(\xi) \geq \dfrac{n}{2-\log 2} = 0.765\cdots n$.

Our strategy: "THE MORE, THE MERRIER." (14th-century, poem *Pearl*, line 850).

$\rightarrow$ Choose $n+1$ linearly independent "good" polynomial approximations in $\mathbb{Z}[X]_{\leq n}$.

Framework: *parametric geometry of numbers*.

Back to Wirsing's problem...

We choose l.i. polynomials $P_1, \dots, P_{n+1}$ which **realize the successive minima** of the symmetric convex body

$$\mathcal{C}_\xi(q) = \Big\{ R \in \mathbb{R}[X]_{\leq n} \,;\, \|R\| \leq 1 \quad \text{and} \quad |R(\xi)| \leq e^{-q} \Big\}$$

with respect to the lattice $\mathbb{Z}[X]_{\leq n}$ (for a good choice of q).

Back to Wirsing's problem...

We choose l.i. polynomials $P_1, \dots, P_{n+1}$ which **realize the successive minima** of the symmetric convex body

$$\mathcal{C}_\xi(q) = \left\{ R \in \mathbb{R}[X]_{\leq n} \,;\, \|R\| \leq 1 \quad \text{and} \quad |R(\xi)| \leq e^{-q} \right\}$$

with respect to the lattice $\mathbb{Z}[X]_{\leq n}$ (for a good choice of q).

Remark. Polynomials realizing *the first minium* = *best polynomial approximations*... ("given" by Dirichlet's Theorem.).

Back to Wirsing's problem...

We choose l.i. polynomials $P_1, \dots, P_{n+1}$ which **realize the successive minima** of the symmetric convex body

$$\mathcal{C}_\xi(q) = \left\{ R \in \mathbb{R}[X]_{\le n} \,;\, \|R\| \le 1 \quad \text{and} \quad |R(\xi)| \le e^{-q} \right\}$$

with respect to the lattice $\mathbb{Z}[X]_{\le n}$ (for a good choice of q).

Remark. Polynomials realizing *the first minium* $=$ *best polynomial approximations*... ("given" by Dirichlet's Theorem.).

We have

- $\|P_1\| \le \cdots \le \|P_{n+1}\|$;
- $|P_k(\xi)| \le |P_1(\xi)|$ (after small correction);
- Control of the norms ?

Back to Wirsing's problem...

We choose l.i. polynomials $P_1, \dots, P_{n+1}$ which **realize the successive minima** of the symmetric convex body

$$\mathcal{C}_\xi(q) = \Big\{ R \in \mathbb{R}[X]_{\leq n} \, ; \, \|R\| \leq 1 \quad \text{and} \quad |R(\xi)| \leq e^{-q} \Big\}$$

with respect to the lattice $\mathbb{Z}[X]_{\leq n}$ (for a good choice of q).

Remark. Polynomials realizing *the first minium* = *best polynomial approximations*... ("given" by Dirichlet's Theorem.).

We have

- $\|P_1\| \leq \cdots \leq \|P_{n+1}\|$;
- $|P_k(\xi)| \leq |P_1(\xi)|$ (after small correction);
- Control of the norms ? $\rightarrow$ Minkowski's second Theorem :

$$|P_1(\xi)| \|P_2\| \cdots \|P_{n+1}\| \asymp 1.$$

Sketch of the proof

Write $\|P_k\| = \|P_2\|^{a_k}$ and $|P_1(\xi)| \asymp \|P_2\|^{-x}$.

Sketch of the proof

Write $\|P_k\| = \|P_2\|^{a_k}$ and $|P_1(\xi)| \asymp \|P_2\|^{-x}$.

Properties of the parameters :

- $1 = a_2 \leq a_3 \leq \cdots \leq a_{n+1}$,
- $a_2 + \cdots + a_{n+1} = x \geq n$ (Minkowski's second theorem),

Sketch of the proof

Write $\|P_k\| = \|P_2\|^{a_k}$ and $|P_1(\xi)| \asymp \|P_2\|^{-x}$.

Properties of the parameters :

- $1 = a_2 \leq a_3 \leq \cdots \leq a_{n+1}$,
- $a_2 + \cdots + a_{n+1} = x \geq n$ (Minkowski's second theorem),

Step 1. Bound from above

$$1 \leq |\det(P_1, \ldots, P_{n+1})| = \left| \begin{array}{cccc} P_1(\xi) & P_2(\xi) & \cdots & P_{n+1}(\xi) \\ P_1'(\xi) & P_2'(\xi) & \cdots & P_{n+1}'(\xi) \\ \hline \mathcal{O}(\|P_1\|) & \mathcal{O}(\|P_2\|) & \cdots & \mathcal{O}(\|P_{n+1}\|) \\ & & & \\ \mathcal{O}(\|P_1\|) & \mathcal{O}(\|P_2\|) & \cdots & \mathcal{O}(\|P_{n+1}\|) \end{array} \right|.$$

Sketch of the proof

$$1 \leq \Big| \underbrace{\det(P_1, \ldots, P_{n+1})}_{\text{involves } P_i(\xi),\, P_i'(\xi)} \Big|.$$

- $\Rightarrow$ at least one of the $P_i'(\xi)$ is "not too" small;
- $\Rightarrow$ at least one of the P_i has a root "rather close" to ξ;
- $\Rightarrow$ (after some computation)

$$\omega_n^*(\xi) \geq \beta_1(x, a_2, \ldots, a_{n+1}) = \frac{x}{a_{n+1}}.$$

Sketch of the proof

Step 2. Get rid of P_{n+1}. Problem : we do not have a basis anymore...

Sketch of the proof

Step 2. Get rid of P_{n+1}. Problem : we do not have a basis anymore...

$\rightarrow$ change the ambient space and work in $\mathbb{R}[X]_{n+1}$.

Sketch of the proof

Step 2. Get rid of P_{n+1}. Problem : we do not have a basis anymore...

$\rightarrow$ change the ambient space and work in $\mathbb{R}[X]_{n+1}$.
$\rightarrow$ Advantage: we can consider $P_1, XP_1, \ldots, P_n, XP_n$ ($2n$ polynomials).

Sketch of the proof

Step 2. Get rid of P_{n+1}. Problem : we do not have a basis anymore...

$\rightarrow$ change the ambient space and work in $\mathbb{R}[X]_{n+1}$.
$\rightarrow$ Advantage: we can consider $P_1, XP_1, \dots, P_n, XP_n$ ($2n$ polynomials).

Form a basis $\mathcal{B}$ of $\mathbb{R}[X]_{n+1}$ by choosing among

$$\{\underbrace{P_1, XP_1, P_2, XP_2}_{start}, \dots, P_i, XP_i, \dots, P_n, XP_n\}$$

(always possible if P_1 and P_2 coprime).

Sketch of the proof

Step 2. Get rid of P_{n+1}. Problem : we do not have a basis anymore...

$\rightarrow$ change the ambient space and work in $\mathbb{R}[X]_{n+1}$.
$\rightarrow$ Advantage: we can consider $P_1, XP_1, \dots, P_n, XP_n$ ($2n$ polynomials).

Form a basis $\mathcal{B}$ of $\mathbb{R}[X]_{n+1}$ by choosing among

$$\{\underbrace{P_1, XP_1, P_2, XP_2}_{start}, \dots, P_i, XP_i, \dots, P_n, XP_n\}$$

(always possible if P_1 and P_2 coprime). Bounding from above

$$1 \leq \underbrace{|\det(\mathcal{B})|}_{\text{involves } P_i(\xi),\ P_i'(\xi)}$$

we obtain $\omega_n^*(\xi) \geq \beta_2(x, a_2, \dots, a_{n+1}) = \dfrac{x + a_{n+1} - 2}{a_n}$.

Step 3. Get rid of P_{n+1} and P_n and repeat the process with a basis of $\mathbb{R}[X]_{n+2}$ choosing among

$$\{\underbrace{P_1, XP_1, X^2P_1, P_2, XP_2, X^2P_2}_{\text{start}}, \dots, P_i, XP_i, X^2P_i, \dots\}.$$

We deduce that, for some explicit function β_3,

$$\omega_n^*(\xi) \geq \beta_3(x, a_2, \dots, a_{n+1}).$$

Sketch of the proof

$\vdots$

Step n. Get rid of $P_3, \dots, P_{n+1}$ and consider the basis of $\mathbb{R}[X]_{2n-1}$

$$\mathcal{B} = \{P_1, XP_1, \dots, X^{n-1}, P_2, XP_2, \dots, X^{n-1}P_2\}$$

Sketch of the proof

$\vdots$

Step n**.** Get rid of $P_3, \ldots, P_{n+1}$ and consider the basis of $\mathbb{R}[X]_{2n-1}$

$$\mathcal{B} = \{P_1, XP_1, \ldots, X^{n-1}, P_2, XP_2, \ldots, X^{n-1}P_2\}$$

Then (as in Wirsing's proof), we have

$$\det(\mathcal{B}) = \mathrm{Res}(P_1, P_2)$$

Sketch of the proof

$\vdots$

Step n**.** Get rid of $P_3, \dots, P_{n+1}$ and consider the basis of $\mathbb{R}[X]_{2n-1}$

$$\mathcal{B} = \{P_1, XP_1, \dots, X^{n-1}, P_2, XP_2, \dots, X^{n-1}P_2\}$$

Then (as in Wirsing's proof), we have

$$\det(\mathcal{B}) = \mathrm{Res}(P_1, P_2)$$

We find a last lower bound

$$\omega_n^*(\xi) \geq \beta_n(x, a_2, \dots, a_{n+1}).$$

Sketch of the proof

Combining all the previous estimates, we get

$$\omega_n^*(\xi) \geq \max\left\{\beta_1, \ldots, \beta_n\right\} =: F(x, a_2, \ldots, a_{n+1}).$$

Sketch of the proof

Combining all the previous estimates, we get

$$\omega_n^*(\xi) \geq \max\left\{\beta_1, \dots, \beta_n\right\} =: F(x, a_2, \dots, a_{n+1}).$$

Final step. Determine $\min F$ on the set of $(x, a_2, \dots, a_{n+1})$ with

$$1 = a_2 \leq \cdots \leq a_{n+1} \quad \text{and} \quad a_2 + \cdots + a_{n+1} = x \geq n.$$

Sketch of the proof

Combining all the previous estimates, we get

$$\omega_n^*(\xi) \geq \max\left\{\beta_1, \dots, \beta_n\right\} =: F(x, a_2, \dots, a_{n+1}).$$

Final step. Determine $\min F$ on the set of $(x, a_2, \dots, a_{n+1})$ with

$$1 = a_2 \leq \cdots \leq a_{n+1} \quad \text{and} \quad a_2 + \cdots + a_{n+1} = x \geq n.$$

We finally find

$$\omega_n^*(\xi) \geq \min F \geq \frac{n}{2 - \log 2}.$$

Thank you for your attention.

Parametric geometry of numbers

Schmidt and Summerer (2009, 2013), Roy (2015)

Basic idea

Setting : consider

- The lattice $\Lambda = \mathbb{Z}^{n+1} \simeq \mathbb{Z}[X]_{\leq n}$;
- A family of symmetric convex bodies $\mathcal{C}(q)$ of $\mathbb{R}^{n+1} \simeq \mathbb{R}[X]_{\leq n}$ (depending on a parameter $q \geq 0$).

Study the *successive minima* associated to those convex bodies w.r.t. Λ.

Recall that the j-th minimum $\lambda_j(q)$ is the minimum of the $\lambda \geq 0$ s.t.

$$\lambda\mathcal{C}(q) \cap \Lambda$$

contains at least j linearly independent polynomials ($j = 1, \dots, n+1$).

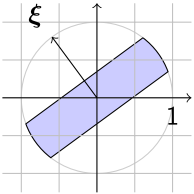
We have $\lambda_1(q) \leq \cdots \leq \lambda_{n+1}(q)$ and **Minkowski's second theorem**:

$$\mathrm{vol}(\mathcal{C}(q))\lambda_1(q)\cdots\lambda_{n+1}(q) \asymp_n 1.$$

Parametric geometry of numbers

Choice of the family ? $\rightarrow$ related to Diophantine problems:

$$\mathcal{C}_\xi(q) = \left\{ R \in \mathbb{R}[X]_{\leq n} \,;\, \|R\| \leq 1 \quad \text{and} \quad |R(\xi)| \leq e^{-q} \right\}.$$

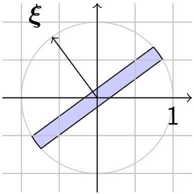


where $\boldsymbol{\xi} = (1, \xi, \xi^2, \dots, \xi^n)$ and we identify $\mathbb{R}^{n+1} \simeq \mathbb{R}[X]_{\leq n}$.

Parametric geometry of numbers

Choice of the family ? $\rightarrow$ related to Diophantine problems:

$$\mathcal{C}_\xi(q) = \left\{ R \in \mathbb{R}[X]_{\leq n} \,;\, \|R\| \leq 1 \quad \text{and} \quad |R(\xi)| \leq e^{-q} \right\}.$$



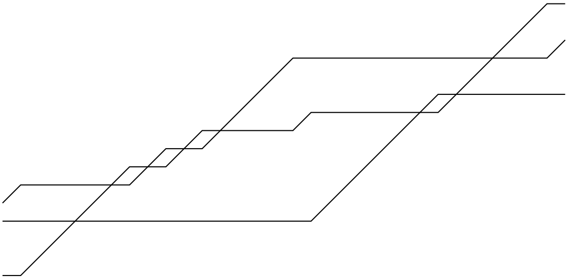
where $\boldsymbol{\xi} = (1, \xi, \xi^2, \dots, \xi^n)$ and we identify $\mathbb{R}^{n+1} \simeq \mathbb{R}[X]_{\leq n}$.

A nice setting

The functions $L_i(q) = \log \lambda_i(q)$ have nice properties :

- $L_1(q) \leq \cdots \leq L_{n+1}(q)$ are continuous,
- L_i piecewise linear with slope 0 or 1,
- $L_1(q) + \cdots + L_{n+1}(q) = q + \mathcal{O}(1)$ (Minkowski's second theorem),
- $\cdots$

Example of the union of the graphs of the L_i ($n = 2$)

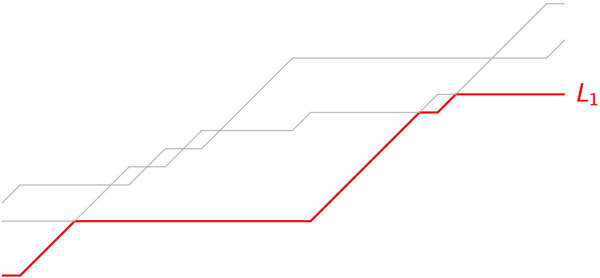


A nice setting

The functions $L_i(q) = \log \lambda_i(q)$ have nice properties :

- $L_1(q) \leq \cdots \leq L_{n+1}(q)$ are continuous,
- L_i piecewise linear with slope 0 or 1,
- $L_1(q) + \cdots + L_{n+1}(q) = q + \mathcal{O}(1)$ (Minkowski's second theorem),
- $\cdots$

Example of the union of the graphs of the L_i ($n = 2$)

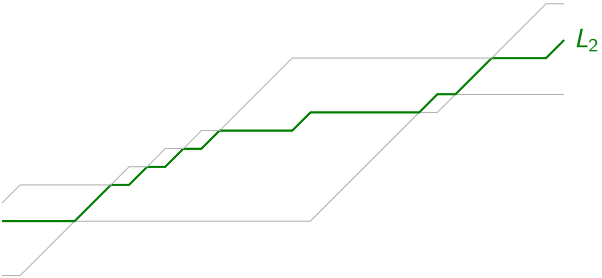


A nice setting

The functions $L_i(q) = \log \lambda_i(q)$ have nice properties :

- $L_1(q) \leq \cdots \leq L_{n+1}(q)$ are continuous,
- L_i piecewise linear with slope 0 or 1,
- $L_1(q) + \cdots + L_{n+1}(q) = q + \mathcal{O}(1)$ (Minkowski's second theorem),
- $\ldots$

Example of the union of the graphs of the L_i ($n = 2$)



A nice setting

The functions $L_i(q) = \log \lambda_i(q)$ have nice properties :

- $L_1(q) \leq \cdots \leq L_{n+1}(q)$ are continuous,
- L_i piecewise linear with slope 0 or 1,
- $L_1(q) + \cdots + L_{n+1}(q) = q + \mathcal{O}(1)$ (Minkowski's second theorem),
- $\cdots$

Example of the union of the graphs of the L_i ($n = 2$)