SIMULTANEOUS RATIONAL APPROXIMATION TO SUCCESSIVE POWERS OF A REAL NUMBER

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Abstract. We develop new tools leading, for each integer $n \geq 4$, to a significantly improved upper bound for the uniform exponent of rational approximation $\widehat{\lambda}_n(\xi)$ to successive powers $1, \xi, \dots, \xi^n$ of a given real transcendental number ξ. As an application, we obtain a refined lower bound for the exponent of approximation to ξ by algebraic integers of degree at most $n+1$. The new lower bound is $n/2 + a\sqrt{n} + 4/3$ with $a = (1 - \log(2))/2 \simeq 0.153$, instead of the current $n/2 + \mathcal{O}(1)$.

1. Introduction

In their seminal 1969 paper [8], Davenport and Schmidt introduce a novel approach to study how well a given real number ξ may be approximated by algebraic integers of degree at most $n+1$, for a given positive integer n. Using geometry of numbers, they show that if, for some $c > 0$ and $\lambda > 0$, there are arbitrarily large values of X for which the conditions

$$|x_0| \leq X \quad \text{and} \quad \max_{1 \leq k \leq n} |x_0 \xi^k - x_k| \leq c X^{-\lambda} \tag{1.1}$$

admit no non-zero integer solution $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$, then for some $c' > 0$ there are infinitely many algebraic integers α of degree at most $n+1$ such that

$$|\xi - \alpha| \leq c' H(\alpha)^{-1/\lambda - 1}$$

where $H(\alpha)$ stands for the height of α, namely the largest absolute value of the coefficients of its irreducible polynomial over $\mathbb{Z}$. Assuming that ξ is not itself an algebraic number of degree at most n, they further show that admissible values for λ are $\lambda = 1$ if $n = 1$, $\lambda = (-1+\sqrt{5})/2$ if $n = 2$, $\lambda = 1/2$ if $n = 3$ and $\lambda = 1/\lfloor n/2 \rfloor$ if $n \geq 4$. Since then, their results have been extended to many other settings which include approximation to ξ by algebraic integers of a given degree [6], by algebraic units of a given degree [27] and by conjugate algebraic integers [19]. A p-adic analog is given in [28] and an extension to a variety of inhomogeneous problems is proposed in [4]. Refined values for λ have also been established by Laurent [10], by Schleischitz [22, 23] and by Badziahin [1].

For each $\xi \in \mathbb{R}$ and each integer $n \geq 1$, let $\widehat{\lambda}_n(\xi)$ (resp. $\lambda_n(\xi)$) denote the supremum of all $\lambda \geq 0$ such that, for $c = 1$, the conditions (1.1) admit a non-zero integer solution

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$\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ for each sufficiently large X (resp. for arbitrarily large values of X). Further, let $\tau_{n+1}(\xi)$ denote the supremum of all $\tau \geq 0$ for which there exist infinitely many algebraic integers α of degree at most $n+1$ with $|\xi - \alpha| \leq H(\alpha)^{-\tau}$. Then, in crude form, the observation of Davenport and Schmidt is that

$$\tau_{n+1}(\xi) \geq \widehat{\lambda}_n(\xi)^{-1} + 1. \tag{1.2}$$

Thus any upper bound on $\widehat{\lambda}_n(\xi)$ yields a lower bound on $\tau_{n+1}(\xi)$.

Assume now that ξ is not itself an algebraic number of degree at most n, namely that $1, \xi, \dots, \xi^n$ are linearly independent over $\mathbb{Q}$, or equivalently that $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$. Then a result of Dirichlet [24, §II.1, Theorem 1A] yields $1/n \leq \widehat{\lambda}_n(\xi)$ and, expecting the equality, it is natural to conjecture that $\tau_{n+1}(\xi) \geq n+1$ as in [25, §5]. For $n = 1$, we have $\widehat{\lambda}_1(\xi) = 1$ and the conjectured lower bound $\tau_2(\xi) \geq 2$ follows. However, for $n = 2$, the upper bound $\widehat{\lambda}_2(\xi) \leq (-1+\sqrt{5})/2 \cong 0.618$ from [8, Theorem 1a] is best possible by [15, Theorem 1.1], and the corresponding lower bound $\tau_3(\xi) \geq (3+\sqrt{5})/2 \cong 2.618$ is also best possible by [16, Theorem 1.1] (see also [20]). This disproves the natural conjecture for $n = 2$ and suggests that it might be false as well for each $n \geq 3$. Any counterexample ξ would have $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$ and $\widehat{\lambda}_n(\xi) > 1/n$. So, it would be transcendental over $\mathbb{Q}$ according to Schmidt's subspace theorem [24, §VI.1, Corollary 1E]. Although the existence of such number remains an open problem for $n \geq 3$, we know by contrast large families of transcendental real numbers ξ with $\widehat{\lambda}_2(\xi) > 1/2$. In chronological order, they are the extremal numbers ξ of [15], the Sturmian continued fractions of [5], Fischler's numbers from [9], the Fibonacci type numbers of [17] and the Sturmian type numbers of [13], all contained in the very general class of numbers studied in [14]. In particular, we know by [17, Corollary] that the values $\widehat{\lambda}_2(\xi)$ with ξ real and transcendental form a dense subset of the interval $[1/2, (-1+\sqrt{5})/2]$.

For $n \geq 3$, recent progresses have been made on upper bounds for $\widehat{\lambda}_n(\xi)$. The estimates $\widehat{\lambda}_3(\xi) \leq 1/2$ and $\widehat{\lambda}_n(\xi) \leq 1/\lfloor n/2 \rfloor$ for $n \geq 4$ from [8, Theorems 2a and 4a] have been refined by Laurent [10] to $\widehat{\lambda}_n(\xi) \leq 1/\lceil n/2 \rceil$ for each $n \geq 3$, together with an important simplification in the proof. When $n = 3$, the best computed upper bound (yet not optimal) remains that of [18],

$$\widehat{\lambda}_3(\xi) \leq \alpha = 0.4245\cdots, \tag{1.3}$$

where α is the root of the polynomial $1 - 3x + 4x^3 - x^4$ in the interval $[1/3, 1/2]$. For even integers $n = 2m \geq 4$, Schleischitz [22, 23] refined the upper bound $\widehat{\lambda}_n(\xi) \leq 1/m$ by reducing to the case where $\lambda_n(\xi) \leq 1/m$ and then by using a transference inequality of Marnat and Moshchevitin [11] relating $\lambda_n(\xi)$ and $\widehat{\lambda}_n(\xi)$. Nevertheless, all those refinements, including those of the recent preprint of Baziahin [1], are of the form $\widehat{\lambda}_n(\xi) \leq 1/(n/2 + c_n)$ with $0 < c_n < 1$. Our main result below improves significantly on this when n is large.

Theorem 1.1. *For any integer $n \geq 2$ and any $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$, we have*

$$\widehat{\lambda}_n(\xi) \leq \frac{1}{n/2 + a\sqrt{n} + 1/3}. \tag{1.4}$$

where $a = (1 - \log(2))/2 \cong 0.1534$.

Note that the multiplicative constant a in the denominator is not optimal and could be improved with additional work. The same applies to the additive constant $1/3$ given the actual choice of a. In view of (1.2), this gives

$$\tau_{n+1}(\xi) \geq n/2 + a\sqrt{n} + 4/3$$

for the same n and ξ.

As explained by Bugeaud in [3, Prop. 3.3], the arguments of Davenport and Schmidt leading to (1.2) can also be adapted to Wirsing's problem of approximating real numbers ξ by algebraic numbers, yielding $\omega_n^*(\xi) \geq \widehat{\lambda}_n(\xi)^{-1}$ for any integer $n \geq 1$, where $\omega_n^*(\xi)$ denotes the supremum of all $\omega > 0$ for which there exist infinitely many algebraic numbers α of degree at most n with $|\xi - \alpha| \leq H(\alpha)^{-\omega-1}$. Thus when $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$, the inequality (1.4) implies that $\omega_n^*(\xi) \geq n/2 + a\sqrt{n} + 1/3$. However, this is superseded by the recent breakthrough of Badziahin and Schleischitz who showed in [2] that $\omega_n^*(\xi) > n/\sqrt{3}$ when $[\mathbb{Q}(\xi) : \mathbb{Q}] > n \geq 4$. Previous to their work, the best lower bounds for large values of n were of the form $n/2 + \mathcal{O}(1)$.

For small values of n, namely for n odd with $5 \leq n \leq 49$ and for n even with $4 \leq n \leq 100$, we obtain the following estimates which improve on Theorem 1.1.

Theorem 1.2. *Suppose that $n = 2m+1 \geq 5$ is odd. Then for each $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$, we have*

$$\widehat{\lambda}_{2m+1}(\xi) \leq \alpha_m,$$

where α_m is the positive root of the polynomial $P_m(x) = 1 - (m+1)x - mx^2$.

Theorem 1.3. *Suppose that $n = 2m \geq 4$ is even. Then for each $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$, we have*

$$\widehat{\lambda}_{2m}(\xi) \leq \beta_m$$

where β_m is the positive root of the polynomial

$$Q_m(x) = \begin{cases} 1 - mx - mx^2 - m(m-1)x^3 & \text{if } m \geq 3, \\ 1 - 3x + x^2 - 2x^3 - 2x^4 & \text{if } m = 2. \end{cases}$$

Again, these upper bounds could be slightly improved with more work, at least for each $n \geq 6$ (even or odd). Note that they are relatively close to $1/(m+2)$ as one finds

$$\frac{1}{m+2} < \alpha_m < \frac{1}{m+2} + \frac{2}{(m+2)^3} \quad \text{and} \quad \frac{1}{m+2} < \beta_m < \frac{1}{m+2} + \frac{7}{(m+2)^3}$$

for each $m \geq 2$. The table below shows how they compare to those of Laurent (L.) for odd $n \leq 13$ and to those of Schleischitz (S.) and Badziahin (B.) for even $n \leq 12$.

n	**L.**	**S.**	**B.**	**new**
4		$0.3706\cdots$	$0.3660\cdots$	$0.3370\cdots$
5	$0.3333\cdots$			$0.2807\cdots$
6		$0.2681\cdots$	$0.2637\cdots$	$0.2444\cdots$
7	$0.2500\cdots$			$0.2152\cdots$
8		$0.2107\cdots$	$0.2071\cdots$	$0.1919\cdots$
9	$0.2000\cdots$			$0.1753\cdots$
10		$0.1737\cdots$	$0.1708\cdots$	$0.1587\cdots$
11	$0.1666\cdots$			$0.1483\cdots$
12		$0.1478\cdots$	$0.1454\cdots$	$0.1357\cdots$
13	$0.1428\cdots$			$0.1286\cdots$

FIGURE 1. Upper bounds for $\widehat{\lambda}_n$

For the proof we develop new tools. The main one concerns the behavior of the function $f(\ell) = \dim \mathcal{U}^\ell(A)$ for $\ell \in \{0, 1, \ldots, n+1\}$ where A is any subspace of $\mathbb{R}^{n+1}$ and $\mathcal{U}^\ell(A)$ stands for the subspace of $\mathbb{R}^{n-\ell+1}$ spanned by the images of A through the projections $(x_0, \ldots, x_n) \mapsto (x_k, \ldots, x_{k+n-\ell})$ for $k = 0, \ldots, \ell$, with the convention that $\mathcal{U}^{n+1}(A) = 0$. In Sections 3 and 4, we show that such a function is concave and monotone increasing as long as $\mathcal{U}^\ell(A) \neq \mathbb{R}^{n-\ell+1}$. We also study the degenerate cases where $f(\ell) < \dim(A) + \ell \leq n - \ell + 1$.

In Section 5, we form a sequence of minimal points $(\mathbf{x}_i)_{i\geq 0}$ for ξ in $\mathbb{Z}^{n+1}$, and recall how the exponents $\widehat{\lambda}_n(\xi)$ and $\lambda_n(\xi)$ can be computed from this data. Given integers $0 \leq j, \ell \leq n$, we say that Property $\mathcal{P}(j, \ell)$ holds if, for any subspace $A = \langle \mathbf{x}_i, \ldots, \mathbf{x}_q \rangle$ of dimension at most $j + 1$ spanned by consecutive minimal points with a large enough initial index i, we have $\dim \mathcal{U}^\ell(A) \geq \dim(A) + \ell$. This is a crucial notion with the remarkable feature that $\mathcal{P}(j, \ell)$ implies $\mathcal{P}(j + 1, \ell - 1)$ when $\ell \geq 1$.

In Sections 6 and 7, we establish some consequences of Properties $\mathcal{P}(0, \ell)$ and $\mathcal{P}(1, \ell)$ respectively and we provide lower bounds on $\widehat{\lambda}_n(\xi)$ which ensure that these properties hold. In Section 6, we also study the general situation where $\mathcal{P}(j, \ell - 1)$ holds but not $\mathcal{P}(j, \ell)$ for some integer $\ell \geq 1$. In Sections 8 and 9, we provide two types of upper bounds for the height of $\mathcal{U}^\ell(A)$ when $\mathcal{P}(j, \ell)$ holds and $A = \langle \mathbf{x}_i, \ldots, \mathbf{x}_q \rangle$ has dimension $j + 1$. In particular, the estimate of Section 9 yields a strong constraint on the growth of the norms $X_i = \|\mathbf{x}_i\|$. These tools are combined in Section 10 to prove Theorem 1.1. Finally Theorems 1.2 and 1.3 are proved respectively in Sections 12 and 13, with the help of a new construction presented in Section 11.

We start in the next section by fixing some notation, including our notion of height for the subspaces of $\mathbb{R}^m$ defined over $\mathbb{Q}$.

2. Heights

For each integer $m \geq 1$, we view $\mathbb{R}^m$ as an Euclidean space for the usual scalar product of points $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ written $\mathbf{x} \cdot \mathbf{y}$, and we denote by $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ the Euclidean norm of a point $\mathbf{x} \in \mathbb{R}^m$. For each integer $k = 1, \dots, m$, we also identify $\bigwedge^k \mathbb{R}^m$ with $\mathbb{R}^{\binom{m}{k}}$ via a choice of ordering of the Plücker coordinates and we denote by $\|\boldsymbol{\alpha}\|$ the resulting Euclidean norm of a point $\boldsymbol{\alpha} \in \bigwedge^k \mathbb{R}^m$.

For any subset A of $\mathbb{R}^m$, we denote by $\langle A \rangle$ the linear subspace of $\mathbb{R}^m$ spanned by A over $\mathbb{R}$. When A is a finite set $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$, we simply write $\langle \mathbf{x}_1, \dots, \mathbf{x}_k \rangle$.

Let V be an arbitrary vector subspace of $\mathbb{R}^m$ defined over $\mathbb{Q}$. If $V \neq 0$, we define its height $H(V)$ as the covolume of the lattice $V \cap \mathbb{Z}^m$ inside V. Explicitly, if $\dim(V) = k$ and if $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a basis of $V \cap \mathbb{Z}^m$ over $\mathbb{Z}$, then

$$H(V) = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\|. \tag{2.1}$$

For $V = 0$, we set $H(0) = 1$. Then, we have the duality relation

$$H(V) = H(V^\perp) \tag{2.2}$$

where $V^\perp$ denotes the orthogonal complement of V in $\mathbb{R}^m$ [26, Chapter I, §8]. In particular, this gives $H(\mathbb{R}^m) = H(0^\perp) = 1$. If $\mathbf{x} \in \mathbb{Z}^m$ is *primitive*, namely if the gcd of its coordinates is 1, we have $H(\langle \mathbf{x} \rangle) = H(\langle \mathbf{x} \rangle^\perp) = \|\mathbf{x}\|$. We will also need the following important inequality of Schmidt

$$H(U \cap V) H(U + V) \leq H(U) H(V), \tag{2.3}$$

valid for any subspaces U and V of $\mathbb{R}^m$ defined over $\mathbb{Q}$ [26, Chapter I, Lemma 8A].

Finally, given $\xi \in \mathbb{R}$ and $\mathbf{x} = (x_0, \dots, x_m) \in \mathbb{Z}^{m+1} \setminus \{0\}$, we define

$$L_\xi(\mathbf{x}) = \max_{1 \leq j \leq m} |x_0 \xi^j - x_j|,$$

and note that

$$L_\xi(\mathbf{x}) \asymp \|\Xi_m \wedge \mathbf{x}\| \quad \text{where} \quad \Xi_m = (1, \xi, \dots, \xi^m)$$

with implicit constants depending only on ξ and m. The latter relation is instructive since the product $\|\Xi_m \wedge \mathbf{x}\| \, \|\Xi_m\|^{-1} \|\mathbf{x}\|^{-1}$ represents the sine of the angle between Ξ_m and $\mathbf{x}$. We will repeatedly use the following generalization of [8, Lemma 9].

Lemma 2.1. *Suppose that $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{Z}^{m+1}$ are linearly independent. Then*

$$H(\langle \mathbf{x}_1, \dots, \mathbf{x}_k \rangle) \leq \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \ll \sum_{i=1}^{k} \|\mathbf{x}_i\| \prod_{j \neq i} L_\xi(\mathbf{x}_j)$$

with an implied constant which depends only on ξ and m.

This follows from (2.1) by writing $\mathbf{x}_j = x_{j,0}\Xi_m + \Delta_j$ for $j = 1, \dots, k$, where $x_{j,0}$ stands for the first coordinate of $\mathbf{x}_j$, and then by expanding the exterior product upon noting that $\|\Delta_j\| \asymp L_\xi(\mathbf{x}_j)$.

3. Three crucial propositions

Let ℓ, n be integers with $0 \le \ell \le n$. For each $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, we denote by

$$\mathbf{x}^{(k,\ell)} = (x_k, \dots, x_{k+n-\ell}) \in \mathbb{R}^{n+1-\ell} \quad (0 \le k \le \ell),$$

the points consisting of $n + 1 - \ell$ consecutive coordinates of $\mathbf{x}$, and we denote by

$$\mathcal{U}^\ell(\mathbf{x}) = \langle\, \mathbf{x}^{(0,\ell)}, \dots, \mathbf{x}^{(\ell,\ell)} \rangle \subseteq \mathbb{R}^{n+1-\ell},$$

the vector subspace of $\mathbb{R}^{n+1-\ell}$ which they generate. In general, for each non-empty subset A of $\mathbb{R}^{n+1}$, we define

$$\mathcal{U}^\ell(A) = \sum_{\mathbf{x}\in A} \mathcal{U}^\ell(\mathbf{x}) \quad \text{and} \quad \mathcal{U}^{n+1}(A) = 0.$$

Then, we have $\mathcal{U}^k(\mathcal{U}^{\ell-k}(A)) = \mathcal{U}^\ell(A) = \mathcal{U}^\ell(\langle A\rangle)$ for any integers $0 \le k \le \ell \le n + 1$.

Our interest in the truncated points $\mathbf{x}^{(k,\ell)}$ comes from the fact that, when $\mathbf{x} \in \mathbb{Z}^{n+1}$ and $\ell < n$, they belong to $\mathbb{Z}^{n+1-\ell}$ and, for given $\xi \in \mathbb{R}$, they satisfy

$$\|\mathbf{x}^{(k,\ell)}\| \le \|\mathbf{x}\| \quad \text{and} \quad L_\xi(\mathbf{x}^{(k,\ell)}) \ll L_\xi(\mathbf{x}) \tag{3.1}$$

with implied constants depending only on ξ and n. So Lemma 2.1 yields

$$H(\mathcal{U}^\ell(\mathbf{x})) \ll \|\mathbf{x}\| L_\xi(\mathbf{x})^{d-1} \quad \text{if} \quad d = \dim \mathcal{U}^\ell(\mathbf{x}) > 0. \tag{3.2}$$

In general, when A is a subspace of $\mathbb{R}^{n+1}$ defined over $\mathbb{Q}$, the subspace $\mathcal{U}^\ell(A)$ of $\mathbb{R}^{n+1-\ell}$ is also defined over $\mathbb{Q}$ and, as the above example shows, we need some information on its dimension in order to estimate its height.

In this section, we state three propositions concerning $\mathcal{U}^\ell(A)$ as a function of ℓ, for a fixed subspace A of $\mathbb{R}^{n+1}$, but postpone their proofs to the next section. In order to state the first one, we recall that a function $f : \{0, \dots, n + 1\} \to \mathbb{R}$ is *convex* if it satisfies the following equivalent conditions

(1) $f(i) - f(i - 1) \le f(i + 1) - f(i)$ for $i = 1, \dots, n$;

(2) $\dfrac{f(j) - f(i)}{j - i} \le \dfrac{f(k) - f(j)}{k - j}$ whenever $0 \le i < j < k \le n + 1$.

We say that f is *concave* if $-f$ is convex. We also fix a positive integer n.

Proposition 3.1. *Let A be a subspace of $\mathbb{R}^{n+1}$. Then $f(\ell) = \dim \mathcal{U}^\ell(A)$ is a concave function of $\ell \in \{0, \dots, n+1\}$. Moreover, there is an integer $m \in \{0, \dots, n+1\}$ for which the function f is monotonically increasing on $\{0, \dots, m\}$, while strictly decreasing with $f(\ell) = n - \ell + 1$ for $\ell \in \{m, \dots, n + 1\}$.*

Figure 2 illustrates this result. Taking it for granted, we deduce a useful corollary.

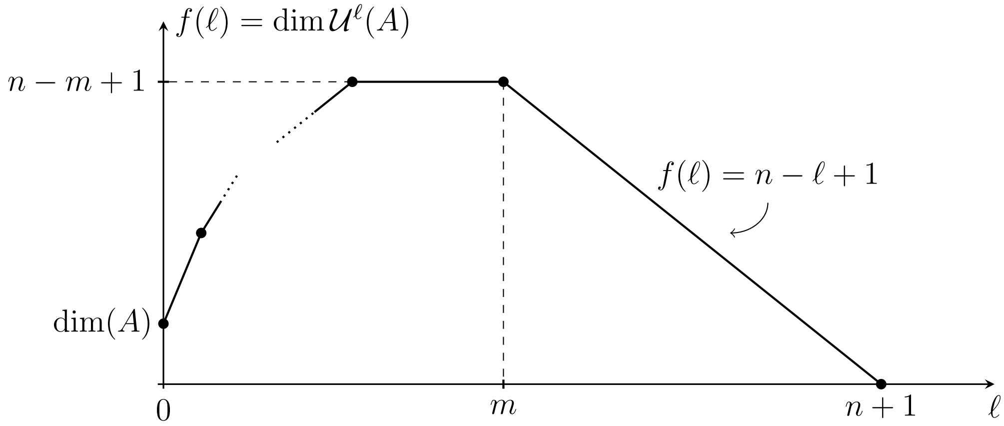

FIGURE 2. Graph of the piecewise linear function interpolating the values $f(\ell) = \dim \mathcal{U}^\ell(A)$ at integers $\ell \in \{0, \dots, n+1\}$.

Corollary 3.2. *Let A be a subspace of $\mathbb{R}^{n+1}$, and let $\ell \in \{1, \dots, n\}$. Then*

(i) $\min\{\dim \mathcal{U}^\ell(A), \dim(A) + \ell - 1\} \le \dim \mathcal{U}^{\ell-1}(A)$;
(ii) $\min\{\dim \mathcal{U}^{\ell-1}(A), n - \ell + 1\} \le \dim \mathcal{U}^\ell(A)$.

Proof. Let f and m be as in Proposition 3.1. If $f(\ell-1) = n-\ell+2$, then $\ell - 1 \ge m$, so $f(\ell) = n - \ell + 1$ and we are done. Otherwise the function f is concave and monotonically increasing on $\{0, \dots, \ell\}$, hence $f(\ell-1) \le f(\ell)$, so (ii) holds. If $f(\ell-1) = f(\ell)$ or if $\ell = 1$, then (i) also holds since $f(0) = \dim \mathcal{U}^0(A) = \dim(A)$. So we may further assume that $f(\ell-1) < f(\ell)$ and that $\ell > 1$. By concavity of f, we deduce that

$$\frac{f(\ell-1) - f(0)}{\ell - 1} \ge f(\ell) - f(\ell-1) \ge 1,$$

hence $f(\ell-1) \ge f(0) + \ell - 1 = \dim(A) + \ell - 1$, and (i) holds again. □

The second proposition provides additional information in the degenerate situation where $\dim \mathcal{U}^\ell(A) < \dim(A) + \ell$.

Proposition 3.3. *Let $j, \ell \ge 0$ be integers with $j + 2\ell \le n$, and let A be a subspace of $\mathbb{R}^{n+1}$ of dimension $j+1$ defined over $\mathbb{Q}$. Suppose that*

$$d := \dim \mathcal{U}^\ell(A) \le j + \ell,$$

and set $V = \mathcal{U}^{n-d}(A)$. Then, we have $0 \le d - j - 1 < \ell \le n - d$ and

$$\dim \mathcal{U}^t(A) = d \quad \text{and} \quad H(\mathcal{U}^t(A)) \asymp H(V)^{n-d-t+1}$$

for each $t = d-j-1, \dots, n-d$, with implied constants depending only on n. Moreover, for such t and for $\mathbf{x} \in \mathbb{R}^{n+1}$, the condition $\mathcal{U}^t(\mathbf{x}) \subseteq \mathcal{U}^t(A)$ is equivalent to $\mathcal{U}^{n-d}(\mathbf{x}) \subseteq V$, thus independent of t.

The last result exhibits the generic behavior of a family of linear maps.

Proposition 3.4. *Let $\ell \in \{0, \dots, n\}$ and let V be a subspace of $\mathbb{R}^{n-\ell+1}$. Suppose that a vector subspace A of $\mathbb{R}^{n+1}$ satisfies $\dim(A) \le n - \ell + 1$ and $\mathcal{U}^\ell(A) \not\subseteq V$. Then, there exists a point $\mathbf{a} = (a_0, \dots, a_\ell) \in \mathbb{Z}^{\ell+1}$ with $\sum_{k=0}^{\ell} |a_k| \le (n+1)^\ell$ such that the linear map*

$$\begin{array}{rccl} \tau_{\mathbf{a}}\colon \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R}^{n-\ell+1} \\ \mathbf{x} & \longmapsto & \sum_{k=0}^{\ell} a_k \mathbf{x}^{(k,\ell)} \end{array} \tag{3.3}$$

is injective on A with $\tau_{\mathbf{a}}(A) \not\subseteq V$.

4. Proofs of the three propositions

Our goal is to prove the statements of the preceding section by re-interpreting them in a polynomial setting similar to that of [19]. In particular, we will connect the function f of Proposition 3.1 to the Hilbert-Samuel function of a graded module over a polynomial ring in two variables. To this end, we start by fixing some notation.

Let $E = \mathbb{R}[T_0, T_1]$ denote the ring of polynomials in two variables T_0 and T_1 over $\mathbb{R}$, and let $D = \mathbb{R}[\delta_0, \delta_1]$ denote the subring of $\mathrm{End}_{\mathbb{R}}(E)$ spanned by the partial derivatives

$$\delta_0 = \frac{\partial}{\partial T_0} \quad \text{and} \quad \delta_1 = \frac{\partial}{\partial T_1}$$

restricted to E. It is easily seen that these commuting linear operators are algebraically independent over $\mathbb{R}$. Thus, D is a commutative ring isomorphic to E. In particular, both D and E are graded rings (by the degree) as well as unique factorization domains. Moreover, E is a D-module for the natural action of the differential operators of D on E.

For each integer $n \ge 0$, we denote by $E_n = \mathbb{R}[T_0, T_1]_n$ and $D_n = \mathbb{R}[\delta_0, \delta_1]_n$ the homogeneous parts of E and D of degree n. We also denote by $\psi_n\colon \mathbb{R}^{n+1} \to E_n$ the linear isomorphism sending a point $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ to the polynomial

$$\psi_n(\mathbf{x}) = \sum_{i=0}^{n} \binom{n}{i} x_i T_0^{n-i} T_1^i.$$

When $n \ge 1$, we find that

$$\delta_0 \psi_n(\mathbf{x}) = n \psi_{n-1}(\mathbf{x}^{(0,1)}) \quad \text{and} \quad \delta_1 \psi_n(\mathbf{x}) = n \psi_{n-1}(\mathbf{x}^{(1,1)}), \tag{4.1}$$

thus $D_1 \psi_n(\mathbf{x}) = \psi_{n-1}(\mathcal{U}^1(\mathbf{x}))$. We deduce that, for any subspace A of $\mathbb{R}^{n+1}$, we have $D_1 \psi_n(A) = \psi_{n-1}(\mathcal{U}^1(A))$ and so, by induction,

$$D_\ell \psi_n(A) = \psi_{n-\ell}(\mathcal{U}^\ell(A)) \quad \text{for each } \ell \in \{0, \dots, n\}. \tag{4.2}$$

Thus, if we identify $\mathbb{R}^{n+1}$ with E_n for each $n \ge 0$, then $\mathcal{U}^\ell(A)$ becomes simply $D_\ell A$ for each subspace A of E_n and each $\ell = 0, \dots, n+1$, including $\ell = n+1$ because $D_{n+1} A = 0$.

From now on, we fix a positive integer n, a subspace A of $\mathbb{R}^{n+1}$, and a spanning set $\{\mathbf{x}_1, \dots, \mathbf{x}_s\}$ of A as a vector space over $\mathbb{R}$. We set $P_i = \psi_n(\mathbf{x}_i)$ for each $i = 1, \dots, s$, and form the D-module homomorphism $\varphi\colon D^s \to E$ given by

$$\varphi(\mathbf{d}) = d_1 P_1 + \cdots + d_s P_s$$

for each $\mathbf{d} = (d_1, \dots, d_s) \in D^s$. Then $M := \ker(\varphi)$ is a graded submodule of D^s. Define

$$f(\ell) = \dim(\varphi(D^s_\ell)) \quad \text{and} \quad g(\ell) = \dim(M_\ell) \quad \text{for each } \ell \in \{0, 1, \dots, n+1\},$$

where $M_\ell = M \cap D^s_\ell$ stands for the homogeneous part of M of degree ℓ, and the dimensions are taken over $\mathbb{R}$. Then, we have

$$f(\ell) = \dim(D^s_\ell) - \dim(M_\ell) = (\ell+1)s - g(\ell) \quad (0 \le \ell \le n+1). \tag{4.3}$$

By (4.2), we also have $\varphi(D^s_\ell) = D_\ell \psi_n(A) = \psi_{n-\ell}(\mathcal{U}^\ell(A))$ for $\ell = 0, \dots, n$. Comparing dimensions, this gives

$$f(\ell) = \dim(\mathcal{U}^\ell(A)) \quad (0 \le \ell \le n+1) \tag{4.4}$$

upon noting that for $\ell = n+1$ both sides vanish. Finally, we define

$$h(\ell) = g(\ell+1) - g(\ell) = \dim(M_{\ell+1}/\delta_1 M_\ell) \quad \text{for each } \ell \in \{0, 1, \dots, n\}. \tag{4.5}$$

With this notation, our main observation is the following.

Lemma 4.1. *For each $\ell \in \{1, \dots, n\}$, we have $h(\ell-1) \le h(\ell)$ with equality if and only if $M_{\ell+1} = D_1 M_\ell$.*

Proof. Fix an integer $\ell \in \{1, \dots, n\}$, and consider the linear map $\nu\colon M_\ell \to M_{\ell+1}/\delta_1 M_\ell$ given by $\nu(\mathbf{d}) = \delta_0 \mathbf{d} + \delta_1 M_\ell$ for each $\mathbf{d} \in M_\ell$. If $\mathbf{d} \in \ker(\nu)$, then $\delta_0 \mathbf{d} = \delta_1 \mathbf{u}$ for some $\mathbf{u} \in M_\ell \subseteq D^s_\ell$. Hence δ_0 divides $\delta_1 \mathbf{u}$ in D^s, so $\mathbf{u} = \delta_0 \mathbf{v}$ for some $\mathbf{v} \in D^s_{\ell-1}$, and then $\mathbf{d} = \delta_1 \mathbf{v}$. Since $\mathbf{d}, \mathbf{u} \in M_\ell$, we find $0 = \varphi(\mathbf{u}) = \delta_0 \varphi(\mathbf{v})$ and $0 = \varphi(\mathbf{d}) = \delta_1 \varphi(\mathbf{v})$, thus $\varphi(\mathbf{v}) = 0$. This means that $\mathbf{v} \in M_{\ell-1}$, and so $\mathbf{d} = \delta_1 \mathbf{v} \in \delta_1 M_{\ell-1}$. This shows that $\ker(\nu) \subseteq \delta_1 M_{\ell-1}$. As the reverse inclusion is clear, we conclude that $\ker(\nu) = \delta_1 M_{\ell-1}$, and so ν induces an injective map from $M_\ell/\delta_1 M_{\ell-1}$ to $M_{\ell+1}/\delta_1 M_\ell$. Comparing dimensions, we deduce that $h(\ell-1) \le h(\ell)$. Moreover, we have the equality $h(\ell-1) = h(\ell)$ if and only if ν is surjective, a condition which amounts to $M_{\ell+1} = \delta_0 M_\ell + \delta_1 M_\ell$ or equivalently to $M_{\ell+1} = D_1 M_\ell$. ☐

As a consequence, we deduce the first assertion of Proposition 3.1.

Corollary 4.2. *The function $g(\ell) = \dim(M_\ell)$ is convex on $\{0, 1, \dots, n+1\}$, while the function $f(\ell) = \dim(\mathcal{U}^\ell(A))$ is concave on the same set.*

Proof. The assertion for g follows directly from the lemma and the definition of h in (4.5). Then (4.3) gives $f(\ell)$ as the sum of two concave functions of ℓ on $\{0, \dots, n+1\}$, namely $(\ell+1)s$ and $-g(\ell)$, thus f is concave. ☐

For each $m \geq 0$, we note that the action of D on E induces a non-degenerate bilinear form

$$\begin{array}{rcl} D_m \times E_m & \longrightarrow & \mathbb{R} \\ (\delta, P) & \longmapsto & \delta P \end{array}$$

which identifies D_m with the dual of E_m, the dual of the natural basis $(T_0^{m-i}T_1^i)_{0\leq i\leq m}$ of E_m being

$$\left(\frac{\delta_0^{m-i}\delta_1^i}{(m-i)!i!}\right)_{0\leq i\leq m}.$$

So, for each subspace W of E_m (resp. W of D_m), its orthogonal space

$$W^\perp = \{\delta \in D_m\,;\, \delta W = 0\} \subseteq D_m \quad \big(\text{resp. } W^\perp = \{P \in E_m\,;\, WP = 0\} \subseteq E_m\big)$$

satisfies $\dim(W^\perp) = m + 1 - \dim(W)$ and $(W^\perp)^\perp = W$. We can now prove the following.

Lemma 4.3. *Let W be a proper subspace of E_m for some $m \geq 0$. Then there are at most m real numbers a for which the differential operator $\delta = \delta_0 + a\delta_1$ is not injective on W.*

Proof. Without loss of generality, we may assume that W has codimension 1 in E_m. Then $W^\perp = \langle\gamma\rangle$ for some non-zero $\gamma \in D_m$. Suppose that $\delta = \delta_0 + a\delta_1 \in D_1$ is not injective on W, for some $a \in \mathbb{R}$. Then δW is a proper subspace of E_{m-1} and so $\beta\delta W = 0$ for some non-zero $\beta \in D_{m-1}$. Therefore $\beta\delta$ belongs to $W^\perp$, and so is proportional to γ. This means that δ divides γ in D. As γ has degree m, it admits at most m non-associate divisors of degree 1. Thus a belongs to a set of at most m numbers. □

The next result is the second part of Proposition 3.1.

Corollary 4.4. *There is a smallest integer $m \in \{0, \dots, n+1\}$ for which $f(m) = n - m + 1$. For this choice of m, the function f is monotonically increasing on $\{0, \dots, m\}$, while strictly decreasing with $f(\ell) = n - \ell + 1$ for $\ell \in \{m, \dots, n+1\}$.*

Proof. The existence of m follows from the fact that $f(n+1) = 0$. For that m, we have $\varphi(D_m^s) = E_{n-m}$. Thus for each $\ell \in \{m, \dots, n+1\}$, we find $\varphi(D_\ell^s) = D_{\ell-m}E_{n-m} = E_{n-\ell}$ and so $f(\ell) = n - \ell + 1$. It remains to prove that f is monotonically increasing on $\{0, \dots, m\}$. This is automatic if $m = 0$. Otherwise, since f is concave on $\{0, \dots, n+1\}$, this amounts to showing that $f(m-1) \leq f(m)$. By the choice of m, the vector space $W = \varphi(D_{m-1}^s)$ is a proper subspace of E_{n-m+1}. Then, by Lemma 4.3, there is some $\delta \in D_1$ which is injective on W, thus $f(m) = \dim(D_1W) \geq \dim(\delta W) = \dim(W) = f(m-1)$. □

Similarly, we will derive Proposition 3.4 from the following result.

Proposition 4.5. *Let $\ell \in \{0, \dots, n\}$ and let S be a subspace of $E_{n-\ell}$. Suppose that a subspace B of E_n satisfies $\dim(B) \leq n - \ell + 1$ and $D_\ell B \not\subseteq S$. Then, there exist $a_0, \dots, a_\ell \in \mathbb{Z}$ with $\sum_{k=0}^{\ell} |a_k| \leq (n+1)^\ell$ such that the differential operator $\delta = \sum_{k=0}^{\ell} a_k\delta_0^{\ell-k}\delta_1^k \in D_\ell$ is injective on B with $\delta B \not\subseteq S$.*

Proof. We proceed by induction on ℓ. For $\ell = 0$, the result is automatic, it suffices to take $a_0 = 1$. Suppose that $\ell \geq 1$ and set $S^* = \{P \in E_{n-\ell+1}\,;\, D_1 P \subseteq S\}$. Then S^* is a subspace of $E_{n-\ell+1}$ and $D_{\ell-1}B \not\subseteq S^*$. So, by induction, we may assume the existence of $a_0^*, \dots, a_{\ell-1}^* \in \mathbb{Z}$ with $\sum_{k=0}^{\ell-1} |a_k^*| \leq (n+1)^{\ell-1}$ for which $\delta^* = \sum_{k=0}^{\ell-1} a_k^* \delta_0^{\ell-1-k}\delta_1^k \in D_{\ell-1}$ is injective on B with $\delta^* B \not\subseteq S^*$. Then, $W = \delta^* B$ is a proper subspace of $E_{n-\ell+1}$ because its dimension is $\dim(B) \leq n - \ell + 1$. Since $W \not\subseteq S^*$, we also have $D_1 W \not\subseteq S$, and so there is at most one $a \in \mathbb{R}$ for which $\gamma = \delta_0 + a\delta_1$ satisfies $\gamma W \subseteq S$. By Lemma 4.3, we can therefore choose $a \in \mathbb{Z}$ with $|a| \leq n - \ell + 1 \leq n$ such that γ is injective on W with $\gamma W \not\subseteq S$. Then $\delta = \gamma\delta^*$ has the required properties. □

To deduce Proposition 3.4, we simply apply the above result with $S = \psi_{n-\ell}(V)$, $B = \psi_n(A)$ and note that, by virtue of (4.1), we have $\delta \circ \psi_n = n(n-1)\cdots(n-\ell+1)\psi_{n-\ell} \circ \tau_{\mathbf{a}}$, with $\mathbf{a} = (a_0, \dots, a_\ell)$.

Finally, for the proof of Proposition 3.3, we need to extend the notion of height on the homogeneous components of D and E. For each $m \geq 0$, we denote by $\psi_m^* \colon \mathbb{R}^{m+1} \to D_m$ the isomorphism which is dual to $\psi_m \colon \mathbb{R}^{m+1} \to E_m$ in the sense that $\psi_m^*(\mathbf{y})\psi_m(\mathbf{x}) = \mathbf{y} \cdot \mathbf{x}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m+1}$. We say that a subspace S of D_m (resp. E_m) is defined over $\mathbb{Q}$ if it is generated by elements of $\mathbb{Q}[\delta_0, \delta_1]_m$ (resp. $\mathbb{Q}[T_0, T_1]_m$) or equivalently if $(\psi_m^*)^{-1}(S)$ (resp. $(\psi_m)^{-1}(S)$ is a subspace V of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$, and we define its height $H(S)$ to be $H(V)$. Then the formula (2.2) translates into $H(S) = H(S^\perp)$ for each subspace S of E_m defined over $\mathbb{Q}$, and its orthogonal space $S^\perp$ in D_m. Moreover, for each non-zero $\delta \in \mathbb{Q}[\delta_0, \delta_1]_m$ and each $k \geq 0$, the subspace $D_k\delta$ of D_{k+m} is defined over $\mathbb{Q}$ of dimension $k+1$ and, by [19, Proposition 5.2], its height satisfies

$$H(D_k\delta) \asymp H(\langle\delta\rangle)^{k+1} \tag{4.6}$$

with implied constants that do not depend on δ.

In this setting, Proposition 3.3 follows immediately from the next result upon setting $B = \psi_n(A)$ and noting that our choice of height yields $H(D_t B) = H(\mathcal{U}^t(A))$ for each $t = 0, \dots, n$.

Proposition 4.6. *Let $j, \ell \geq 0$ be integers with $j + 2\ell \leq n$, and let B be a subspace of E_n of dimension $j+1$ defined over $\mathbb{Q}$. Set $d = \dim(D_\ell B)$ and suppose that $d \leq j + \ell$. Then, we have $0 \leq d - j - 1 < \ell \leq n - d$ and there exists a non-zero operator $\delta \in \mathbb{Q}[\delta_0, \delta_1]_d$ such that*

$$D_t B = (D_{n-d-t}\delta)^\perp, \quad \dim(D_t B) = d \quad \text{and} \quad H(D_t B) \asymp H(\langle\delta\rangle)^{n-d-t+1}$$

for $t = d - j - 1, \dots, n - d$, with implied constants depending only on n. Moreover, for such t and for $P \in E_n$, the condition $D_t P \subseteq D_t B$ is independent of t and amounts to $\delta P = 0$.

Proof. We may assume that $B = \psi_n(A)$ (with A defined over $\mathbb{Q}$), and then $f(t) = \dim(D_t B)$ for $t = 0, \dots, n+1$. Let m be as in Corollary 4.4, so that f is monotonically increasing on

$\{0,\dots,m\}$. Since $f(\ell)=d\le j+\ell<n-\ell+1$, this corollary gives $\ell<m$. We deduce that

$$f(0)=j+1\le f(\ell)=d\le f(m)=n-m+1,$$

thus $1\le d-j$, while the hypotheses yield $d-j\le \ell\le n-d$. In turn this gives

$$f(d-j)\le f(\ell)=f(0)+(d-j)-1.$$

So, f is not strictly increasing on $\{0,\dots,d-j\}$. Being concave, it is therefore constant on $\{d-j-1,\dots,m\}$, equal to $f(\ell)=d$. In particular, we obtain $m=n+1-f(m)=n-d+1$, and so $f(n-d)=d$. This means that $D_{n-d}B$ has codimension 1 in E_d. As it is defined over $\mathbb{Q}$, we deduce that $(D_{n-d}B)^{\perp}=\langle\delta\rangle$ for some non-zero $\delta\in\mathbb{Q}[\delta_0,\delta_1]_d$.

For each $t=d-j-1,\dots,n-d$, the subspace D_tB of E_{n-t} has dimension d, while $D_{n-d-t}\delta$ has codimension d in D_{n-t}. Since their product is

$$(D_{n-d-t}\delta)(D_tB)=\delta D_{n-d}B=0,$$

we deduce that $D_tB=(D_{n-d-t}\delta)^{\perp}$. Thus, using (4.6), we obtain

$$H(D_tB)=H(D_{n-d-t}\delta)\asymp H(\langle\delta\rangle)^{n-d-t+1}.$$

Finally, for $P\in E_n$, the condition $D_tP\subseteq D_tB$ may be rewritten as $(D_{n-d-t}\delta)(D_tP)=0$, so it is equivalent to $\delta P\in D_{n-d}^{\perp}=0$ (inside E_{n-d}). □

5. Minimal points and properties $\mathcal{P}(j,\ell)$

From now on, we fix a positive integer n and a real number ξ with $[\mathbb{Q}(\xi):\mathbb{Q}]>n$. Our goal is to establish an upper bound for $\widehat{\lambda}_n(\xi)$ which depends only on n.

Since $[\mathbb{Q}(\xi):\mathbb{Q}]>n$, non-zero points $\mathbf{x},\mathbf{y}\in\mathbb{Z}^{n+1}$ which satisfy $L_\xi(\mathbf{x})=L_\xi(\mathbf{y})<1$ have a non-zero first coordinate and come by pairs $\mathbf{y}=\pm\mathbf{x}$. Thus, for each large enough real number $X\ge 1$, there is a unique pair of non-zero points $\pm\mathbf{x}$ in $\mathbb{Z}^{n+1}$ with $\|\mathbf{x}\|\le X$ for which $L_\xi(\mathbf{x})<1$ is minimal. We choose the one whose first coordinate is positive and, like Davenport and Schmidt in [8], we call it the *minimal point corresponding to X*. This differs slightly from their own definition, but it plays the same role.

We order these minimal points in a sequence $(\mathbf{x}_i)_{i\ge 0}$ by increasing norm. Then,

- their norms $X_i=\|\mathbf{x}_i\|$ are positive and strictly increasing,
- the quantities $L_i=L_\xi(\mathbf{x}_i)$ are strictly decreasing,
- if $L_\xi(\mathbf{x})<L_i$ for some $i\ge 0$ and some non-zero $\mathbf{x}\in\mathbb{Z}^{n+1}$, then $\|\mathbf{x}\|\ge X_{i+1}$.

In terms of the associated sequences $(X_i)_{i\ge 0}$ and $(L_i)_{i\ge 0}$, we have the well-known formulas

$$\lambda_n(\xi)=\limsup_{i\to\infty}\frac{-\log(L_i)}{\log(X_i)}\quad\text{and}\quad \widehat{\lambda}_n(\xi)=\liminf_{i\to\infty}\frac{-\log(L_i)}{\log(X_{i+1})} \tag{5.1}$$

which follow from the definition of these exponents given in the introduction. In particular, if $\widehat{\lambda}_n(\xi)>\lambda$ for some $\lambda\in\mathbb{R}$, then $L_i=o(X_{i+1}^{-\lambda})$ and a fortiori

$$L_i\ll X_{i+1}^{-\lambda}, \tag{5.2}$$

where from now on all implicit multiplicative constants are independent of i.

By construction, each minimal point $\mathbf{x}_i$ is primitive and so we have

$$H(\langle\, \mathbf{x}_i \rangle) = X_i.$$

For subspaces spanned by two consecutive minimal points, a simple adaptation of the proofs of [7, Lemma 2] and [15, Lemma 4.1] yields the following estimate.

Lemma 5.1. *For each $i \geq 0$, we have $H\big(\langle\, \mathbf{x}_i, \mathbf{x}_{i+1} \rangle\big) = \|\mathbf{x}_i \wedge \mathbf{x}_{i+1}\| \asymp X_{i+1} L_i$.*

More generally, we are interested in the subspaces $\langle \mathbf{x}_i, \dots, \mathbf{x}_q \rangle$ of $\mathbb{R}^{n+1}$ spanned by minimal points with consecutive indices, as in [12, §3] (see also [11]). It is well-known that, for each $i \geq 0$, we have

$$\langle \mathbf{x}_i, \mathbf{x}_{i+1}, \dots \rangle = \sum_{k=i}^{\infty} \langle \mathbf{x}_k \rangle = \mathbb{R}^{n+1} \tag{5.3}$$

because these are subspaces of $\mathbb{R}^{n+1}$ defined over $\mathbb{Q}$ which contain $\lim_{k\to\infty} \|\mathbf{x}_k\|^{-1}\mathbf{x}_k = \|\Xi\|^{-1}\Xi$ where $\Xi = (1, \xi, \dots, \xi^n)$ has $\mathbb{Q}$-linearly independent coordinates. This justifies the following construction.

Definition 5.2. For each $i \geq 0$ and each $j = 0, \dots, n-1$, we set

$$\sigma_j(i) = q, \quad A_j(i) = \langle \mathbf{x}_i, \dots, \mathbf{x}_q \rangle \quad \text{and} \quad Y_j(i) = X_{q+1}$$

where $q \geq i$ is the largest index for which $\dim\langle \mathbf{x}_i, \dots, \mathbf{x}_q \rangle = j+1$. We also set

$$A_n(i) = \mathbb{R}^{n+1} \quad \text{and} \quad Y_{-1}(i) = X_i.$$

So, for $j = 0, \dots, n-1$, we have

$$\dim(A_j(i)) = j+1 \quad \text{and} \quad A_{j+1}(i) = \langle \mathbf{x}_i, \dots, \mathbf{x}_q, \mathbf{x}_{q+1} \rangle \quad \text{where } q = \sigma_j(i).$$

For $j = 0$, we note that $\sigma_0(i) = i$, $A_0(i) = \langle \mathbf{x}_i \rangle$ and $Y_0(i) = X_{i+1}$. For $j \geq 1$, the following notation is useful.

Definition 5.3. We denote by I the set of indices $i \geq 1$ such that $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. We say that $i < j$ are *consecutive elements* of I or that j is the *successor* of i in I if j is the smallest element of I with $j > i$.

When $n = 1$, the set I is empty. However, when $n > 1$, we may form $q = \sigma_j(i)$ for each $i \geq 1$ and each $j = 1, \dots, n-1$. Since $\langle\, \mathbf{x}_{q-1}, \mathbf{x}_q \rangle \subseteq A_j(i)$ and $\mathbf{x}_{q+1} \notin A_j(i)$, we deduce that $q \in I$. Thus I is infinite. Moreover, if $i < j$ are consecutive elements of I, we have

$$\langle\, \mathbf{x}_i, \mathbf{x}_{i+1} \rangle = \cdots = \langle\, \mathbf{x}_{j-1}, \mathbf{x}_j \rangle \neq \langle\, \mathbf{x}_j, \mathbf{x}_{j+1} \rangle,$$

Applying Lemma 5.1, we obtain the following useful estimate.

Lemma 5.4. *Suppose that $n \geq 2$. Then I is an infinite set and for each pair $i < j$ of consecutive elements of I, we have $X_j L_{j-1} \asymp X_{i+1} L_i$.*

The above also shows that, for each integer $i \geq 0$, we have $\sigma_1(i) = j$ where j is the smallest element of I with $j > i$, so $A_1(i) = A_1(j-1)$ and $A_2(i) = A_2(j-1) = \langle \mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1} \rangle$.

In the next sections, we will provide upper bound estimates for the height of the subspaces $\mathcal{U}^\ell(A_j(i))$ when the following condition is fulfilled.

Definition 5.5. Let $j, \ell \in \{0, \dots, n\}$. We say that property $\mathcal{P}(j, \ell)$ holds if, for each sufficiently large integer $i \geq 0$ and each $m = 0, \dots, j$, we have $\dim \mathcal{U}^\ell(A_m(i)) \geq m + \ell + 1$.

Of course, this depends on our fixed choice of ξ and n. Clearly $\mathcal{P}(n, 0)$ holds, because $\mathcal{U}^0(A_m(i)) = A_m(i)$ has dimension $m+1$ for each $i \geq 0$ and each $m = 0, \dots, n$. Moreover $\mathcal{P}(j, \ell)$ implies $\mathcal{P}(j-1, \ell)$ if $j > 0$. The next result provides a further crucial implication.

Proposition 5.6. *Suppose that property $\mathcal{P}(j, \ell)$ holds for some $j, \ell \in \{0, \dots, n\}$. Then we have $j + 2\ell \leq n$. If moreover $\ell > 0$, then $\mathcal{P}(j+1, \ell-1)$ holds as well.*

Proof. By hypothesis, $\mathcal{U}^\ell(A_j(i))$ is a subspace of $\mathbb{R}^{n-\ell+1}$ of dimension at least $j + \ell + 1$ for each large enough i. Comparing dimensions yields $j + 2\ell \leq n$. Now, suppose that $\ell > 0$, and set $A = A_m(i)$ for a choice of integers $m \in \{0, \dots, j+1\}$ and $i \geq 0$. By Corollary 3.2(i), we have

$$\min\{\dim \mathcal{U}^\ell(B), m + \ell\} \leq \dim \mathcal{U}^{\ell-1}(A)$$

for any subspace B of A. If $m \leq j$, we choose $B = A_m(i)$. Otherwise, we choose $B = A_{m-1}(i)$. Then, assuming i large enough, we have $\dim \mathcal{U}^\ell(B) \geq \dim(B) + \ell \geq m + \ell$ because of $\mathcal{P}(j, \ell)$, and so $\dim \mathcal{U}^{\ell-1}(A) \geq m + \ell$. Thus $\mathcal{P}(j+1, \ell-1)$ holds. □

6. Property $\mathcal{P}(0, \ell)$

We keep the notation of the preceding section, and fix a real number λ with $0 < \lambda < \widehat{\lambda}_n(\xi)$, thus $\lambda < 1$. In this section, we derive useful consequences of the assumption that, for some $j \geq 0$ and $\ell \geq 1$, we have $\mathcal{P}(j, \ell-1)$ but not $\mathcal{P}(j, \ell)$. As an example of application, we recover an important result of Badziahin and Schleischitz from [2] which yields $\mathcal{P}(0, \ell)$ under a simple condition on ℓ. We recall our convention that all implicit multiplicative constants are independent of i, and we start with two general lemmas.

Lemma 6.1. *Let $0 \leq j, \ell \leq n$ be integers. Suppose that*

$$d := \liminf_{i \to \infty} \dim \mathcal{U}^\ell(A_j(i)) \leq n - \ell.$$

Then there are arbitrarily large integers $i \geq 1$ for which

$$\dim \mathcal{U}^\ell(A_j(i)) = d \quad \textit{and} \quad \mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(A_j(i)). \tag{6.1}$$

Proof. Set $V_i = \mathcal{U}^\ell(A_j(i))$ for each $i \geq 0$, and let E denote the infinite set of integers $i \geq 0$ for which $\dim(V_i) = d$. Choose $p \in E$ large enough so that $\dim(V_i) \geq d$ for each $i \geq p$. Using (5.3), we find

$$\sum_{i=p}^{\infty} V_i = \mathcal{U}^\ell(\langle \mathbf{x}_p, \mathbf{x}_{p+1}, \dots \rangle) = \mathcal{U}^\ell(\mathbb{R}^{n+1}) = \mathbb{R}^{n-\ell+1}.$$

Thus, there exists a smallest $q \in E$ with $q \geq p$ such that $\sum_{i=p}^{q} V_i = \mathbb{R}^{n-\ell+1}$. Since V_p has dimension $d \leq n - \ell$, it is a proper subspace of $\mathbb{R}^{n-\ell+1}$. So, we must have $q > p$, and thus $V_{q-1} \neq V_q$ by the choice of q. As $\dim(V_{q-1}) \geq d = \dim(V_q)$, this means that $V_{q-1} \not\subseteq V_q$ and therefore (6.1) holds for $i = q > p$. □

For any integer $m \geq 1$, any subspace of $\mathbb{R}^m$ defined over $\mathbb{Q}$ has height at least 1. The next lemma provides an instance where this lower bound can be improved.

Lemma 6.2. *Let $\ell \in \{0, \dots, n\}$ and let V be a subspace of $\mathbb{R}^{n+1-\ell}$ defined over $\mathbb{Q}$. Suppose that $\mathcal{U}^\ell(\mathbf{x}_i) \subseteq V$ and that $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq V$ for some $i \geq 1$. Then, $1 \ll H(V) L_{i-1}$.*

Proof. Choose $m \in \{0, \dots, \ell\}$ such that $\mathbf{y} := \mathbf{x}_{i-1}^{(m,\ell)} \notin V$ and set $U = \langle \mathbf{y}, \mathbf{z} \rangle$ where $\mathbf{z} := \mathbf{x}_i^{(m,\ell)}$. Then, we have $U \cap V = \langle \mathbf{z} \rangle$ and so $H(U \cap V) = g^{-1}\|\mathbf{z}\| \asymp g^{-1} X_i$ where g denotes the gcd of the coordinates of $\mathbf{z}$. Since $\mathbf{y}$ and $g^{-1}\mathbf{z}$ are integer points of U, we also find

$$H(U) \leq \|\mathbf{y} \wedge g^{-1}\mathbf{z}\| = g^{-1}\|\mathbf{y} \wedge \mathbf{z}\| \ll g^{-1} X_i L_{i-1},$$

thus $1 \leq H(U + V) \ll H(U \cap V)^{-1} H(U) H(V) \ll H(V) L_{i-1}$. □

We now come to the main result of this section.

Proposition 6.3. *Let $j \geq 0$ and $\ell \geq 1$ be integers with $j + 2\ell \leq n$. Suppose that $\mathcal{P}(j, \ell - 1)$ holds but not $\mathcal{P}(j, \ell)$. Then, there are arbitrarily large values of $i \geq 1$ for which*

$$\dim \mathcal{U}^\ell(A_j(i)) = \ell + j \quad \textit{and} \quad \mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(A_j(i)) \subsetneq \mathbb{R}^{n-\ell+1}. \tag{6.2}$$

For those i, we further have

$$1 \ll H\big(\mathcal{U}^{\ell-1}(A_j(i))\big) L_{i-1}^{n-j-2\ell+2}. \tag{6.3}$$

Proof. Since $\mathcal{P}(j, \ell - 1)$ holds, Corollary 3.2(ii) gives

$$\dim \mathcal{U}^\ell(A_j(i)) \geq \min\{\ell + j,\, n - \ell + 1\} = \ell + j$$

for each sufficiently large i. Since $\mathcal{P}(j, \ell)$ does not hold, we conclude that

$$\liminf_{i \to \infty} \dim \mathcal{U}^\ell(A_j(i)) = \ell + j \leq n - \ell.$$

Then, Lemma 6.1 provides infinitely many i for which (6.2) holds. For such i, Proposition 3.3 applies with $A = A_j(i)$, $d = \ell + j$ and $t \in \{\ell - 1, \ell\}$. Setting $V = \mathcal{U}^{n-\ell-j}(A)$, it gives

$$H(\mathcal{U}^{\ell-1}(A)) \asymp H(V)^{n-j-2\ell+2} \quad \text{and} \quad \mathcal{U}^{n-\ell-j}(\mathbf{x}_{i-1}) \not\subseteq V.$$

Since $\mathcal{U}^{n-\ell-j}(\mathbf{x}_i) \subseteq V$, Lemma 6.2 yields $1 \ll H(V) L_{i-1}$. Then (6.3) follows. □

The following restatement of [2, Lemma 3.1] is central to the present paper.

Proposition 6.4 (Badziahin-Schleischitz, 2021)**.** *Suppose that $\widehat{\lambda}_n(\xi) > 1/(n-\ell+1)$ for some integer ℓ with $0 \le \ell \le n/2$. Then $\mathcal{P}(0,\ell)$ holds. More precisely, we have $\dim \mathcal{U}^\ell(\mathbf{x}_i) = \ell+1$ for each sufficiently large i .*

The proof given in [2] is based on a method of Laurent from [10]. To illustrate Proposition 6.3, we give the following alternative argument.

Proof. Since $\mathcal{P}(0,0)$ holds, there is a largest integer $m \ge 0$ for which $\mathcal{P}(0,m)$ holds. Suppose that $m < \ell$. Since $\mathcal{P}(0,m+1)$ does not hold and $2(m+1) \le n$, Proposition 6.3 shows the existence of arbitrarily large values of i for which

$$1 \ll H(\mathcal{U}^m(\mathbf{x}_i)) L_{i-1}^{n-2m}.$$

As (3.2) gives $H(\mathcal{U}^m(\mathbf{x}_i)) \ll X_i L_i^m$, this yields $1 \ll X_i L_{i-1}^{n-m}$. Then, by the formulas (5.1), we deduce that $\widehat{\lambda}_n(\xi) \le 1/(n-m) \le 1/(n-\ell+1)$. This contradiction shows that $\mathcal{P}(0,\ell)$ holds. The second assertion of the lemma follows since $\dim \mathcal{U}^\ell(\mathbf{x}_i) \le \ell+1$ for each $i \ge 0$. □

If $\mathcal{P}(0,\ell)$ holds for some $\ell \ge 1$, then $\dim \mathcal{U}^\ell(\mathbf{x}_i) = \ell+1$ for each large enough i and, for these i, we obtain $1 \le H(\mathcal{U}^\ell(\mathbf{x}_i)) \ll X_i L_i^\ell \ll X_i X_{i+1}^{-\ell\lambda}$ by (3.2) and our choice of λ. This implies that $\lambda \le 1/\ell$ and so $\widehat{\lambda}_n(\xi) \le 1/\ell$. Combined with Proposition 6.3, this observation yields

$$\widehat{\lambda}_n(\xi) \le \max\{1/(n-\ell+1), 1/\ell\}$$

for each integer ℓ with $1 \le \ell \le n/2$. Thus, if $n \ge 2$, we obtain $\widehat{\lambda}_n(\xi) \le 1/\lfloor n/2 \rfloor$, which is the estimate of Davenport and Schmidt from [8] mentioned in the introduction. We conclude with another consequence of $\mathcal{P}(0,\ell)$.

Lemma 6.5. *Suppose that $\mathcal{P}(0,\ell)$ holds for some integer $\ell \ge 1$ with $2\ell < n$. Then, there are infinitely many $i \ge 1$ for which $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i)$ and, for those i, we have*

$$X_{i+1}^\vartheta \ll X_i \quad \textit{where} \quad \vartheta = \frac{\ell\lambda}{1-\lambda}$$

Proof. Since $\mathcal{P}(0,\ell)$ holds, we have $\dim \mathcal{U}^\ell(\mathbf{x}_i) = \ell+1 \le n-\ell$ for each sufficiently large i. By Lemma 6.1, we deduce that both $\dim \mathcal{U}^\ell(\mathbf{x}_i) = \ell+1$ and $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i)$ for infinitely many $i \ge 1$. For those i, we have $H(\mathcal{U}^\ell(\mathbf{x}_i)) \ll X_i L_i^\ell$ by (3.2), and Lemma 6.2 gives

$$1 \ll H(\mathcal{U}^\ell(\mathbf{x}_i)) L_{i-1} \ll X_i L_i^\ell L_{i-1} \ll X_i^{1-\lambda} X_{i+1}^{-\ell\lambda},$$

so $X_{i+1}^\vartheta \ll X_i$. □

7. Property $\mathcal{P}(1,\ell)$

With the notation of the preceding sections (including the choice of λ), we provide below a sufficient condition on ℓ and λ for property $\mathcal{P}(1,\ell)$ to hold. We also establish consequences of $\mathcal{P}(1,\ell)$. The results of this section provide a first step towards the proof of Theorems 1.2 and 1.3. We start with a crude height estimate.

Lemma 7.1. *If $\mathcal{P}(1,\ell)$ holds for some $\ell\ge 0$, then $H(\mathcal{U}^\ell(\mathbf{x}_i,\mathbf{x}_{i+1}))\ll X_{i+1}^{1-(\ell+1)\lambda}$.*

Proof. For given $i\ge 0$ and $\ell\in\{0,\dots,n\}$, the subspace $\mathcal{U}^\ell(\mathbf{x}_i,\mathbf{x}_{i+1})$ of $\mathbb{R}^{n-\ell+1}$ is generated by the set $\{\mathbf{x}_i^{(0,\ell)},\dots,\mathbf{x}_i^{(\ell,\ell)},\mathbf{x}_{i+1}^{(0,\ell)},\dots,\mathbf{x}_{i+1}^{(\ell,\ell)}\}$ which consists of integer points $\mathbf{y}$ with $\|\mathbf{y}\|\le X_{i+1}$ and $L_\xi(\mathbf{y})\ll L_i$. If $\mathcal{P}(1,\ell)$ holds and i is large enough, this space has dimension $d\ge\ell+2$, and so Lemma 2.1 gives $H(\mathcal{U}^\ell(\mathbf{x}_i,\mathbf{x}_{i+1}))\ll X_{i+1}L_i^{d-1}\le X_{i+1}^{1-(\ell+1)\lambda}$. □

By the above, property $\mathcal{P}(1,\ell)$ implies that $\lambda\le 1/(\ell+1)$ and so $\widehat{\lambda}_n(\xi)\le 1/(\ell+1)$. The next lemma provides finer estimates.

Lemma 7.2. *Suppose that $n\ge 2$ and that $\mathcal{P}(1,\ell)$ holds for some integer $\ell\ge 0$. Then, for each pair of consecutive elements $i<j$ of I, we have*

$$H(\mathcal{U}^\ell(\mathbf{x}_i,\mathbf{x}_{i+1}))\ll X_{j+1}^{-\ell\lambda}X_{i+1}^{1-\lambda}\quad\textit{and}\quad H(\mathcal{U}^\ell(\mathbf{x}_{i-1},\mathbf{x}_i,\mathbf{x}_{i+1}))\ll X_{j+1}^{-\ell\lambda}X_{i+1}^{1-\lambda}X_i^{-e\lambda},$$

where $e=\dim\mathcal{U}^\ell(\mathbf{x}_{i-1},\mathbf{x}_i,\mathbf{x}_{i+1})-\ell-2$.

Proof. For a pair of consecutive elements $i<j$ of I, we have $\langle\,\mathbf{x}_i,\mathbf{x}_{i+1}\rangle=\langle\,\mathbf{x}_{j-1},\mathbf{x}_j\rangle$. Thus we have a chain of subspaces

$$\begin{aligned}U=\mathcal{U}^\ell(\mathbf{x}_j)\subseteq V&=\mathcal{U}^\ell(\mathbf{x}_i,\mathbf{x}_{i+1})=U+\mathcal{U}^\ell(\mathbf{x}_{j-1})\\ \subseteq W&=\mathcal{U}^\ell(\mathbf{x}_{i-1},\mathbf{x}_i,\mathbf{x}_{i+1})=V+\mathcal{U}^\ell(\mathbf{x}_{i-1}).\end{aligned}$$

If i is large enough, then by property $\mathcal{P}(1,\ell)$ we have $\dim(U)=\ell+1$, $\dim(V)=\dim(U)+a$ and $\dim(W)=\dim(V)+b$ for some integers $a\ge1$ and $b\ge 0$. As each subspace $\mathcal{U}^\ell(\mathbf{x}_h)$ is generated by integer points $\mathbf{y}$ with $\|\mathbf{y}\|\le X_h$ and $L_\xi(\mathbf{y})\ll L_h$, Lemma 2.1 gives

$$H(V)\ll X_jL_j^\ell L_{j-1}^a\quad\text{and}\quad H(W)\ll X_jL_j^\ell L_{j-1}^aL_{i-1}^b.$$

Finally, by Lemma 5.4, we have $X_jL_{j-1}\asymp X_{i+1}L_i$. Since $a\ge1$, we deduce that

$$H(V)\ll L_j^\ell X_jL_{j-1}\asymp L_j^\ell X_{i+1}L_i\quad\text{and}\quad H(W)\ll L_j^\ell X_jL_{j-1}L_{i-1}^e\asymp L_j^\ell X_{i+1}L_iL_{i-1}^e,$$

where $e=a+b-1=\dim(W)-\ell-2\ge0$. The conclusion follows. □

Under the hypotheses of Lemma 7.2, we have $1\le H(\mathcal{U}^\ell(\mathbf{x}_i,\mathbf{x}_{i+1}))\ll X_{j+1}^{-\ell\lambda}X_{i+1}^{1-\lambda}$ and so $X_{j+1}^\vartheta\ll X_{i+1}$ with $\vartheta=\ell\lambda/(1-\lambda)$, for each pair of consecutive elements $i<j$ of I. A fortiori, this implies that $X_{i+1}^\vartheta\ll X_i$ for each $i\ge0$. The following result yields a weaker estimate but assumes $\mathcal{P}(1,\ell-1)$ instead of $\mathcal{P}(1,\ell)$.

Lemma 7.3. *Suppose that $\mathcal{P}(1,\ell-1)$ holds for some integer ℓ with $1\le \ell\le n/2$, and that $\widehat{\lambda}_n(\xi)>1/(2\ell)$. Then, for each $i\ge 0$, we have $X_iL_i^\ell\gg 1$ and so $X_{i+1}^{\ell\lambda}\ll X_i$.*

Proof. By definition of property $\mathcal{P}(1,\ell-1)$ we have,

$$\dim \mathcal{U}^{\ell-1}(\mathbf{x}_{i-1},\mathbf{x}_i)\ge \ell+1 \quad\text{and}\quad \dim \mathcal{U}^{\ell-1}(\mathbf{x}_i)\ge \ell$$

for each sufficiently large $i\ge 1$. Fix such an integer i. Then, Corollary 3.2(ii) gives

$$\dim \mathcal{U}^{\ell}(\mathbf{x}_i)\ge \min\{\dim \mathcal{U}^{\ell-1}(\mathbf{x}_i), n-\ell+1\}\ge \ell.$$

If $\dim \mathcal{U}^{\ell}(\mathbf{x}_i)\ge \ell+1$, then $1\le H(\mathcal{U}^{\ell}(\mathbf{x}_i))\ll X_iL_i^\ell$ and we are done. Otherwise, $\mathcal{U}^{\ell}(\mathbf{x}_i)$ has dimension ℓ. Then, Proposition 3.3 applies with $A=\langle\, \mathbf{x}_i\rangle$, $j=0$, $d=\ell$, $V=\mathcal{U}^{n-\ell}(\mathbf{x}_i)$ and $t=\ell-1$. It gives

$$\dim \mathcal{U}^{\ell-1}(\mathbf{x}_i)=\ell \quad\text{and}\quad H(\mathcal{U}^{\ell-1}(\mathbf{x}_i))\asymp H(V)^{n-2\ell+2}\ge H(V)^2.$$

Moreover, since $\dim \mathcal{U}^{\ell-1}(\mathbf{x}_{i-1},\mathbf{x}_i)>\ell$, we have $\mathcal{U}^{\ell-1}(\mathbf{x}_{i-1})\not\subseteq \mathcal{U}^{\ell-1}(\mathbf{x}_i)$ and so the proposition also gives $\mathcal{U}^{n-\ell}(\mathbf{x}_{i-1})\not\subseteq V$. By Lemma 6.2, this in turn yields $H(V)L_{i-1}\gg 1$, thus we find

$$1\ll H(\mathcal{U}^{\ell-1}(\mathbf{x}_i))L_{i-1}^2\ll X_iL_i^{\ell-1}L_{i-1}^2\ll X_i^{1-2\lambda}L_i^{\ell-1}.$$

As $\widehat{\lambda}_n(\xi)>1/(2\ell)$, we may assume that $\lambda\ge 1/(2\ell)$. This gives $1\ll X_i^{1-1/\ell}L_i^{\ell-1}$ and so $1\ll X_iL_i^\ell$. □

We now come to the main result of this section.

Proposition 7.4. *Suppose that, for some integer $\ell\ge 1$ with $1+2\ell\le n$, property $\mathcal{P}(0,\ell)$ holds but not $\mathcal{P}(1,\ell)$. Then we have $0\le 1-(n-\ell)\lambda-\ell\lambda^2$.*

Proof. Since $\mathcal{P}(0,\ell)$ holds, we have $\ell+1=\dim \mathcal{U}^{\ell}(A_0(i))\le \dim \mathcal{U}^{\ell}(A_1(i))$ for each sufficiently large $i\ge 0$. Since $\mathcal{P}(1,\ell)$ does not hold, we also have $\dim \mathcal{U}^{\ell}(A_1(i))\le \ell+1$ for arbitrarily large values of i. Thus, Lemma 6.1 applies with $j=1$ and $d=\ell+1$, and so there are arbitrarily large integers $i\ge 1$ with

$$\dim \mathcal{U}^{\ell}(A_1(i))=\ell+1 \quad\text{and}\quad \mathcal{U}^{\ell}(\mathbf{x}_{i-1})\not\subseteq \mathcal{U}^{\ell}(A_1(i))\subseteq \mathbb{R}^{n-\ell+1}.$$

For these i, Proposition 3.3 applies with $A=A_1(i)$, $j=1$, $d=\ell+1$ and any $t\in\{\ell-1,\ell\}$. Setting $V=\mathcal{U}^{n-\ell-1}(A_1(i))$, it gives

$$H\big(\mathcal{U}^{\ell-1}(A_1(i))\big)\asymp H(V)^{n-2\ell+1} \quad\text{and}\quad H\big(\mathcal{U}^{\ell}(A_1(i))\big)\asymp H(V)^{n-2\ell}.$$

Since $\mathcal{U}^{\ell}(\mathbf{x}_{i-1})\not\subseteq \mathcal{U}^{\ell}(A_1(i))$, it also gives $\mathcal{U}^{n-\ell-1}(\mathbf{x}_{i-1})\not\subseteq V$. By Lemma 6.2, this implies that $1\ll H(V)L_{i-1}\ll H(V)X_i^{-\lambda}$ and so

$$X_i^\lambda\ll H(V).$$

Because of $\mathcal{P}(0,\ell)$, the subspace $\mathcal{U}^\ell(\mathbf{x}_i)$ of $\mathcal{U}^\ell(A_1(i))$ has dimension $\ell+1$ when i is large enough, and then it coincides with $\mathcal{U}^\ell(A_1(i))$. Moreover, $\mathcal{P}(1,\ell-1)$ holds by Proposition 5.6. Thus using (3.2) and Lemma 7.1, the above estimates imply

$$X_i^{(n-2\ell)\lambda} \ll H(\mathcal{U}^\ell(\mathbf{x}_i)) \ll X_i L_i^\ell \ll X_i X_{i+1}^{-\ell\lambda},$$
$$X_i^{(n-2\ell+1)\lambda} \ll H(\mathcal{U}^{\ell-1}(\mathbf{x}_i,\mathbf{x}_{i+1})) \ll X_{i+1}^{1-\ell\lambda}.$$

The second row of estimates implies $1-\ell\lambda>0$ and provides a lower bound for X_{i+1} in terms of X_i. Substituting it in the first row and comparing powers of X_i, we deduce that

$$(1-\ell\lambda)(n-2\ell)\lambda \le (1-\ell\lambda)-(\ell\lambda)(n-2\ell+1)\lambda,$$

which after simplications reduces to $0\le 1-(n-\ell)\lambda-\ell\lambda^2$. □

Corollary 7.5. *Suppose that $n\ge 3$. Let ℓ be an integer with $1\le \ell<n/2$ and let ρ denote the unique positive root of the polynomial $P(x)=1-(n-\ell)x-\ell x^2$. Then, we have $\rho>1/(n-\ell+1)$. If $\widehat{\lambda}_n(\xi)>\rho$, then $\mathcal{P}(1,\ell)$ holds.*

Proof. Let $k=n-\ell+1$. Since $P(1/k)=(n-2\ell+1)/k^2>0$, we have $\rho>1/k$. If $\widehat{\lambda}_n(\xi)>\rho$, we may assume that $\lambda>\rho$. Then we have $\lambda>1/k=1/(n-\ell+1)$ and thus property $\mathcal{P}(0,\ell)$ holds by Proposition 6.4. Since $\lambda>\rho$, we also have $P(\lambda)<0$, and so the preceding proposition implies that $\mathcal{P}(1,\ell)$ holds. □

We remarked after Lemma 7.1 that property $\mathcal{P}(1,\ell)$ implies $\widehat{\lambda}_n(\xi)\le 1/(\ell+1)$. Thus, with the notation and hypotheses of the above corollary, we obtain

$$\widehat{\lambda}_n(\xi)\le \max\{1/(\ell+1),\rho\}.$$

If $n=2m+1\ge 3$ is odd, we may choose $\ell=m$. Then ρ is the positive root of $P(x)=1-(m+1)x-mx^2$, denoted by α_m in the statement of Theorem 1.2. As $P(1/(m+1))<0$, this yields $\widehat{\lambda}_n(\xi)\le 1/(m+1)=\lceil n/2\rceil^{-1}$ which is the main result of Laurent in [10].

8. First general height estimates

With the notation of the preceding section (including the choice of λ), we first show that property $\mathcal{P}(j,\ell)$ yields special bases for the subspaces $\mathcal{U}^\ell(A_j(i))$. Then, we deduce an upper bound on the height of these subspaces in terms of the quantities $Y_j(i)$ from Definition 5.2.

Lemma 8.1. *Suppose that $\mathcal{P}(j,\ell)$ holds for some integers $j,\ell\in\{0,\dots,n\}$. For each large enough integer $i\ge 0$ and each integer $q\ge i$ such that $A_j(i)=\langle \mathbf{x}_i,\dots,\mathbf{x}_q\rangle$, there exist an integer $e\ge 0$ and a basis $\{\mathbf{y}_0,\mathbf{y}_1,\dots,\mathbf{y}_{\ell+j+e}\}$ of $\mathcal{U}^\ell(A_j(i))$ made of points of $\mathbb{Z}^{n+1-\ell}$ of norm $\le X_q$ with*

$$\begin{cases} L_\xi(\mathbf{y}_m)\ll X_{q+1}^{-\lambda} & \text{for } 0\le m\le \ell,\\ L_\xi(\mathbf{y}_{\ell+m})\ll Y_{j-m}(i)^{-\lambda} & \text{for } 1\le m\le j,\\ L_\xi(\mathbf{y}_{\ell+j+m})\ll Y_0(i)^{-\lambda} & \text{for } 1\le m\le e.\end{cases}$$

Proof. We proceed by induction on j. If $j = 0$, we have $q = i$, and $\mathcal{U}^\ell(A_0(i)) = \mathcal{U}^\ell(\mathbf{x}_i)$ has dimension $\ell + 1$ for i large enough. Then the points $\mathbf{y}_m = \mathbf{x}_i^{(m,\ell)}$ for $m = 0, \dots, \ell$ have the required properties. Now suppose that $j \geq 1$. For i large enough, $\mathcal{U}^\ell(A_j(i))$ has dimension $\geq \ell + j + 1$. Choose $q \geq i$ such that $A_j(i) = \langle \mathbf{x}_i, \dots, \mathbf{x}_q \rangle$, and choose p minimal with $i < p \leq q$ such that $\dim\langle \mathbf{x}_p, \dots, \mathbf{x}_q \rangle = j$ or equivalently such that $A_{j-1}(p) = \langle \mathbf{x}_p, \dots, \mathbf{x}_q \rangle$. Since $\mathcal{P}(j-1, \ell)$ holds, we may assume by induction that, when i is large enough, the vector space $\mathcal{U}^\ell(A_{j-1}(p))$ contains linearly independent points $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{\ell+j-1}$ of $\mathbb{Z}^{n+1-\ell}$ of norm $\leq X_q$ with $L_\xi(\mathbf{y}_m) \ll X_{q+1}^{-\lambda}$ for $0 \leq m \leq \ell$ and $L_\xi(\mathbf{y}_{\ell+m}) \ll Y_{j-1-m}(p)^{-\lambda}$ for $1 \leq m \leq j-1$. Since $\mathbf{x}_{p-1} \notin \langle \mathbf{x}_p, \dots, \mathbf{x}_q \rangle$, we have

$$\dim\langle \mathbf{x}_i, \dots, \mathbf{x}_r \rangle \geq 1 + \dim\langle \mathbf{x}_p, \dots, \mathbf{x}_r \rangle$$

for each $r = p, \dots, q$. Thus, for each integer m with $1 \leq m \leq j-1$, we have $\sigma_m(i) \leq \sigma_{m-1}(p) < q$, and so $Y_m(i) \leq Y_{m-1}(p)$. By the above, this means that $L_\xi(\mathbf{y}_{\ell+m}) \ll Y_{j-m}(i)^{-\lambda}$ for $1 \leq m \leq j-1$. Since $\{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{\ell+j-1}\}$ is a linearly independent subset of $\mathcal{U}^\ell(A_j(i))$, we may complete it to a basis $\{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{\ell+j+e}\}$ for some $e \geq 0$ by adding $e+1$ points of the form $\mathbf{x}_h^{(s,\ell)}$ with $i \leq h \leq q$ and $0 \leq s \leq \ell$. These new points belong to $\mathbb{Z}^{n+1-\ell}$, have norm $\leq X_q$, and satisfy $L_\xi(\mathbf{y}_{\ell+j+m}) \ll L_i \ll Y_0(i)^{-\lambda}$ for $0 \leq m \leq e$. □

Proposition 8.2. *Suppose that $\mathcal{P}(j, \ell)$ holds for some integers $j, \ell \in \{0, \dots, n\}$ with $j \geq 1$. For each large enough $i \geq 0$, we have*

$$H\big(\mathcal{U}^\ell(A_j(i))\big) \ll Y_{j-1}(i)^{1-\ell\lambda}\Big(\prod_{m=1}^{j} Y_{j-m}(i)^{-\lambda}\Big)Y_0(i)^{-e\lambda}$$

with $e = \dim\mathcal{U}^\ell(A_j(i)) - \ell - j - 1 \geq 0$.

Proof. For given $i \geq 0$, choose $q \geq i$ minimal such that $A_j(i) = \langle \mathbf{x}_i, \dots, \mathbf{x}_q \rangle$. Then, assuming i large enough so that $e \geq 0$, consider the basis $\{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{\ell+j+e}\}$ of $\mathcal{U}^\ell(A_j(i))$ provided by Lemma 8.1. By the choice of q, we have $X_q = Y_{j-1}(i) \leq X_{q+1}$ and so $L_\xi(\mathbf{y}_m) \ll Y_{j-1}(i)^{-\lambda}$ for $0 \leq m \leq \ell$. Since this basis consists of integer points, we also have

$$H\big(\mathcal{U}^\ell(A_j(i))\big) \leq \|\mathbf{y}_0 \wedge \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_{\ell+j+e}\|.$$

We conclude by applying Lemma 2.1 along with the estimates of Lemma 8.1. □

When $\ell = 0$, property $\mathcal{P}(j, \ell)$ holds and we obtain the following estimate.

Corollary 8.3. $H(A_j(i)) \ll Y_{j-1}(i) \prod_{m=0}^{j-1} Y_m(i)^{-\lambda}$ *for all $j = 0, \dots, n-1$ and all $i \geq 0$.*

Corollary 8.4. *Suppose that $\mathcal{P}(j, \ell)$ holds for some integers $j \geq 1$ and $\ell \geq 0$ with $j + 2\ell < n$. Then there are arbitrarily large integers $i \geq 1$ for which $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i)$ and*

$$1 \ll Y_{j-1}(i)^{1-\ell\lambda}\Big(\prod_{m=1}^{j} Y_{j-m}(i)^{-\lambda}\Big)X_i^{-\lambda}. \tag{8.1}$$

Proof. Let $d = \liminf_{i\to\infty} \dim \mathcal{U}^\ell(A_j(i))$. If $d = \ell + j + 1$, then we have $d \le n - \ell$ and Lemma 6.1 provides infinitely many $i \ge 1$ for which $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(A_j(i))$. For those i, we have $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i)$ and Lemma 6.2 gives $1 \ll H(V)L_{i-1}$ with $V = \mathcal{U}^\ell(A_j(i))$. Then (8.1) follows using $L_{i-1} \ll X_i^{-\lambda}$ and the upper bound for $H(V)$ provided by Proposition 8.2. Otherwise, for each sufficiently large i, we have $\dim \mathcal{U}^\ell(A_j(i)) > \ell + j + 1$ and (8.1) follows directly from Proposition 8.2 using $H\big(\mathcal{U}^\ell(A_j(i))\big) \ge 1$, $Y_0(i) = X_{i+1} \ge X_i$ and $e \ge 1$. Moreover, as $2\ell < n$, Lemma 6.1 also gives $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i)$ for infinitely many $i \ge 1$. □

9. An alternative height estimate

Keeping the same notation, we derive a second height estimate for $\mathcal{U}^\ell(A_j(i))$ by an indirect process, as in the proof of Lemma 6.2, namely by writing this space as a sum of two subspaces with a well-chosen one dimensional intersection, and then by applying Schmidt's height inequality (2.3).

Proposition 9.1. *Suppose that $\mathcal{P}(j,\ell)$ holds for some integers $1 \le j \le \ell < n$. For each $i \ge 0$, we have*

$$H\big(\mathcal{U}^\ell(A_j(i))\big) \ll H\big(A_j(i)\big) Y_j(i)^{-(\ell-j+1)\lambda} \prod_{m=1}^{j-1} Y_m(i)^{-\lambda}.$$

The main feature of this estimate is that it involves a negative power of $Y_j(i)$ and so, as we will see, it yields an upper bound for $Y_j(i)$ in terms of $Y_0(i), \dots, Y_{j-1}(i)$. The proof requires the following simple observation.

Lemma 9.2. *Let $\tau_{\mathbf{a}} \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n-\ell+1}$ be the linear map given by* (3.3) *for fixed $\ell \in \{0, \dots, n\}$ and $\mathbf{a} = (a_0, \dots, a_\ell) \in \mathbb{Z}^{\ell+1} \setminus \{0\}$. Then we have $\|\tau_{\mathbf{a}}(\mathbf{x}_i)\| \asymp X_i$ as $i \to \infty$.*

Proof. Since $\lim_{i\to\infty} X_i^{-1}\mathbf{x}_i = \|\Xi\|^{-1}\Xi$ where $\Xi = (1, \xi, \dots, \xi^n)$, we find

$$\lim_{i\to\infty} X_i^{-1}\tau_{\mathbf{a}}(\mathbf{x}_i) = \|\Xi\|^{-1}\tau_{\mathbf{a}}(\Xi) = \|\Xi\|^{-1}(a_0 + a_1\xi + \cdots + a_\ell\xi^\ell)(1, \xi, \dots, \xi^{n-\ell}).$$

As $[\mathbb{Q}(\xi) : \mathbb{Q}] > n \ge \ell$, this limit is non-zero, and the conclusion follows. □

Proof of Proposition 9.1. We may assume that i is large enough so that Lemma 8.1 applies. Choose $q \ge i$ maximal such that $A_j(i) = \langle \mathbf{x}_i, \dots, \mathbf{x}_q \rangle$, and to simplify notation set $A = A_j(i)$. Then, by definition, we have $X_{q+1} = Y_j(i)$. Moreover, we have $j + 2\ell \le n$ by $\mathcal{P}(j,\ell)$ (see Proposition 5.6), thus $\dim(A) = j + 1 \le n$, and so A is a proper subspace of $\mathbb{R}^{n+1}$. Using the basis of $\mathcal{U}^\ell(A)$ provided by Lemma 8.1 for the present choice of q, we set

$$W = \langle \mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{\ell+j-1} \rangle \subseteq \mathbb{R}^{n+1-\ell}.$$

Since $\dim(W) = \ell + j < \dim \mathcal{U}^\ell(A)$, we have $\mathcal{U}^\ell(A) \not\subseteq W$. As A is a proper subspace of $\mathbb{R}^{n+1}$, Proposition 3.4 provides a non-zero point $\mathbf{a} = (a_0, \dots, a_\ell) \in \mathbb{Z}^{\ell+1}$ with $\|\mathbf{a}\| \ll 1$ such that the linear map $\tau_{\mathbf{a}}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1-\ell}$ is injective on A with $\tau_{\mathbf{a}}(A) \not\subseteq W$. Thus we have $\dim(\tau_{\mathbf{a}}(A)\cap W) \le j$ and so there are ℓ points $\mathbf{z}_0,\dots,\mathbf{z}_{\ell-1}$ among $\mathbf{y}_0,\mathbf{y}_1,\dots,\mathbf{y}_{\ell+j-1}$ such that

$$\tau_{\mathbf{a}}(A)\cap\langle \mathbf{z}_0,\dots,\mathbf{z}_{\ell-1}\rangle = 0.$$

By construction, these are integer points of norm $\le X_q$ and, since $j\le \ell$, we may order them so that

$$\begin{cases} L_\xi(\mathbf{z}_m) \ll X_{q+1}^{-\lambda} = Y_j(i)^{-\lambda} & \text{for } 0\le m\le \ell-j,\\ L_\xi(\mathbf{z}_{\ell-j+m}) \ll Y_{j-m}(i)^{-\lambda} & \text{for } 1\le m\le j-1.\end{cases} \tag{9.1}$$

Since $\dim\mathcal{U}^\ell(A) = \ell+j+1+e$ for some integer $e\ge 0$, we may complete $\{\mathbf{z}_0,\dots,\mathbf{z}_{\ell-1}\}$ to a maximal linearly independent subset $\{\mathbf{z}_0,\dots,\mathbf{z}_{\ell-1+e}\}$ of $\mathcal{U}^\ell(A)$ such that

$$\mathcal{U}^\ell(A) = \tau_{\mathbf{a}}(A)\oplus\langle \mathbf{z}_0,\dots,\mathbf{z}_{\ell-1+e}\rangle$$

by adding integer points of the form $\mathbf{x}_h^{(s,\ell)}$ with $i\le h\le q$ and $0\le s\le \ell$. These new points have norm $\le X_q$ and satisfy

$$L_\xi(\mathbf{z}_{\ell-1+m}) \ll L_i \ll Y_0(i)^{-\lambda} \quad \text{for } 1\le m\le e. \tag{9.2}$$

Define

$$U = \tau_{\mathbf{a}}(A), \quad \mathbf{z} = \tau_{\mathbf{a}}(\mathbf{x}_q) \quad\text{and}\quad V = \langle \mathbf{z},\mathbf{z}_0,\dots,\mathbf{z}_{\ell-1+e}\rangle,$$

so that

$$U+V = \mathcal{U}^\ell(A) \quad\text{and}\quad U\cap V = \langle \mathbf{z}\rangle.$$

Since $\|\mathbf{a}\|\ll 1$, we find

$$H(U)\ll H(A), \quad \|\mathbf{z}\|\asymp X_q \quad\text{and}\quad L_\xi(\mathbf{z})\ll L_q \ll X_{q+1}^{-\lambda} = Y_j(i)^{-\lambda},$$

where the middle estimate $\|\mathbf{z}\|\asymp X_q$ comes from Lemma 9.2. We deduce that

$$H(U\cap V) = g^{-1}\|\mathbf{z}\| \asymp g^{-1}X_q$$

where g denotes the content of $\mathbf{z}$. Since $g^{-1}\mathbf{z}$ is an integer point, we further have

$$H(V) \le \|g^{-1}\mathbf{z}\wedge\mathbf{z}_0\wedge\cdots\wedge\mathbf{z}_{\ell-1+e}\| = g^{-1}\|\mathbf{z}\wedge\mathbf{z}_0\wedge\cdots\wedge\mathbf{z}_{\ell-1+e}\|.$$

Applying Lemma 2.1 with the estimates (9.1), (9.2), this implies

$$\begin{aligned} H(V) &\ll g^{-1}X_qY_j(i)^{-(\ell-j+1)\lambda}\Big(\prod_{m=1}^{j-1} Y_{j-m}(i)^{-\lambda}\Big)Y_0(i)^{-e\lambda}\\ &\ll H(U\cap V)Y_j(i)^{-(\ell-j+1)\lambda}\prod_{m=1}^{j-1} Y_{j-m}(i)^{-\lambda}.\end{aligned}$$

The conclusion follows since $H(\mathcal{U}^\ell(A)) \ll H(U)H(V)/H(U\cap V)$ by (2.3). □

Combining Proposition 9.1 with the crude estimate $H\big(\mathcal{U}^\ell(A_j(i))\big)\ge 1$ and the upper bound for $H(A_j(i))$ given by Corollary 8.3, we obtain the following upper bound for $Y_j(i)$.

Corollary 9.3. *Suppose that $\mathcal{P}(j,\ell)$ holds for some integers $1 \le j \le \ell < n$. Then, for each $i \ge 0$, we have*

$$Y_j(i)^{(\ell-j+1)\lambda} \ll Y_{j-1}(i)\Big(\prod_{m=1}^{j-1} Y_m(i)^{-2\lambda}\Big)Y_0(i)^{-\lambda}.$$

When $j = 1$, this can be reformulated as follows.

Corollary 9.4. *Suppose that $\mathcal{P}(1,\ell)$ holds for some integer $1 \le \ell < n$. Then the ratio*

$$\vartheta = \frac{\ell\lambda}{1-\lambda} \tag{9.3}$$

satisfies $0 < \vartheta \le 1$ and we have $Y_1(i)^{\vartheta} \ll Y_0(i)$ and $Y_0(i)^{\vartheta} \ll Y_{-1}(i)$ for each $i \ge 0$.

Proof. By Corollary 9.3, we have $Y_1(i)^{\ell\lambda} \ll Y_0(i)^{1-\lambda}$ for each $i \ge 0$. Since $\lambda < 1$, this yields $Y_1(i)^{\vartheta} \ll Y_0(i)$ for all $i \ge 0$ and thus $\vartheta \le 1$. For $i \ge 1$, this in turn gives

$$Y_0(i)^{\vartheta} = X_{i+1}^{\vartheta} \le Y_1(i-1)^{\vartheta} \ll Y_0(i-1) = X_i = Y_{-1}(i).$$ □

More generally, Corollary 9.3 admits the following consequence.

Corollary 9.5. *Suppose that $\mathcal{P}(j,\ell)$ holds for some integers $1 \le j \le \ell < n$, and that $\vartheta^{j-1} + \vartheta^{j} \ge 1$ where ϑ is given by (9.3). Then we have $Y_m(i)^{\vartheta} \ll Y_{m-1}(i)$ for each each $i \ge 0$ and each $m = 0, 1, \dots, j$.*

Proof. Since $\mathcal{P}(1,\ell)$ holds, Corollary 9.4 gives $0 < \vartheta \le 1$ and $Y_m(i)^{\vartheta} \ll Y_{m-1}(i)$ for $m = 0, 1$. So, we are done if $j = 1$. Suppose now that $j \ge 2$. Since $\vartheta \le 1$, the hypothesis $\vartheta^{j-1} + \vartheta^{j} \ge 1$ implies that $\vartheta^{j-2} + \vartheta^{j-1} \ge 1$. Since $\mathcal{P}(j-1,\ell)$ holds, we may assume by induction that $Y_m(i)^{\vartheta} \ll Y_{m-1}(i)$ for each $i \ge 0$ and each $m = 0, \dots, j-1$. Then, using Corollary 9.3, we obtain $Y_j(i)^{(\ell-j+1)\lambda} \ll Y_{j-1}(i)^{\rho}$ where

$$\begin{aligned}\rho &= 1 - 2\lambda\Big(\sum_{m=1}^{j-1} \vartheta^{j-1-m}\Big) - \lambda\vartheta^{j-1}\\ &= 1 - \lambda - \lambda\sum_{m=1}^{j-1}(\vartheta^{m-1} + \vartheta^{m}) \le \frac{\ell\lambda}{\vartheta} - \lambda\sum_{m=1}^{j-1}\frac{1}{\vartheta} = \frac{(\ell-j+1)\lambda}{\vartheta},\end{aligned}$$

and so $Y_j(i)^{\vartheta} \ll Y_{j-1}(i)$. □

10. Main proposition and proof of Theorem 1.1

We first prove the following general statement and then optimize the choice of parameters to deduce Theorem 1.1. The notation, including the choice of λ is as in the preceding sections.

Proposition 10.1. *Let $1 \le k \le \ell$ be integers with $k+2\ell = n$. Suppose that*
$$\frac{1}{2} \le \vartheta^k \le 1 \quad \text{where} \quad \vartheta = \frac{\ell\lambda}{1-\lambda}.$$
Then we have $\lambda \le \eta^{-1}$ where $\eta = \ell + \vartheta + \vartheta^2 + \cdots + \vartheta^{k+1}$.

Proof. Assume on the contrary that $\lambda > \eta^{-1}$. Since $\vartheta \le 1$, we have $\eta \le \ell+k+1 = n-\ell+1$, thus $\lambda > (n-\ell+1)^{-1}$, and so Proposition 6.4 shows that $\mathcal{P}(0,\ell)$ holds. By Lemma 6.5, this implies that $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i)$ for infinitely many $i \ge 1$, and that

$$Y_0(i)^\vartheta = X_{i+1}^\vartheta \ll Y_{-1}(i) = X_i \quad \text{whenever} \quad \mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i). \tag{10.1}$$

Let j be the largest integer with $0 \le j \le k$ for which $\mathcal{P}(j,\ell)$ holds. Since $\vartheta \le 1$, we have $\vartheta^{j-1} + \vartheta^j \ge \vartheta^{k-1} + \vartheta^k \ge 1$. So, if $j \ge 1$, Corollary 9.5 gives

$$Y_m(i)^\vartheta \ll Y_{m-1}(i) \quad \text{for each} \quad m = 0,1,\dots,j \quad \text{and} \quad i \ge 0. \tag{10.2}$$

Suppose that $j < k$. Then $\mathcal{P}(j+1,\ell)$ does not hold. However, by Proposition 5.6, $\mathcal{P}(j+1,\ell-1)$ holds because $\mathcal{P}(j,\ell)$ does. Thus, by Proposition 6.3, there are arbitrarily large $i \ge 1$ for which

$$\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(A_{j+1}(i)) \quad \text{and} \quad 1 \ll H\big(\mathcal{U}^{\ell-1}(A_{j+1}(i))\big) L_{i-1}^{n-j-2\ell+1}.$$

Using Proposition 8.2 to estimate from above the height of $\mathcal{U}^{\ell-1}(A_{j+1}(i))$ and recalling that $n = k+2\ell$ and $L_{i-1} \ll Y_{-1}(i)^{-\lambda}$, this gives

$$1 \ll Y_j(i)^{1-(\ell-1)\lambda} \Big(\prod_{m=0}^{j} Y_{j-m}(i)^{-\lambda} \Big) Y_{-1}(i)^{-(k-j+1)\lambda}.$$

For those i, we have $\mathcal{U}^\ell(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^\ell(\mathbf{x}_i)$. Thus, using (10.1) if $j=0$ and (10.2) else, we find

$$\begin{aligned} 0 &\le 1 - \lambda(\ell-1) - \lambda\Big(\sum_{m=0}^{j} \vartheta^m\Big) - \lambda(k-j+1)\vartheta^{j+1} \\ &\le 1 - \lambda\Big(\ell - 1 + \sum_{m=0}^{k+1} \vartheta^m\Big) = 1 - \lambda\eta, \end{aligned}$$

against the hypothesis that $\lambda > \eta^{-1}$.

The above contradiction shows that $j = k$. Thus $\mathcal{P}(k,\ell)$ holds and so $\mathcal{P}(k+1,\ell-1)$ holds as well (by Proposition 5.6). As $(k+1)+2(\ell-1) = n-1 < n$, Corollary 8.4 provides arbitrarily large values of i for which

$$1 \ll Y_k(i)^{1-(\ell-1)\lambda} \Big(\prod_{m=0}^{k} Y_{k-m}(i)^{-\lambda} \Big) Y_{-1}(i)^{-\lambda}.$$

Since $j = k \ge 1$, (10.2) applies. So, we conclude that

$$0 \le 1 - \lambda\Big(\ell - 1 + \sum_{m=0}^{k+1} \vartheta^m\Big) = 1 - \lambda\eta,$$

which again contradicts the hypothesis that $\lambda > \eta^{-1}$. □

Proof of Theorem 1.1. For $2 \le n \le 3$, the upper bound for $\widehat{\lambda}_n(\xi)$ provided by the theorem is weaker than the prior ones mentioned in the introduction. For $4 \le n \le 11$, they are also weaker than those coming from Theorems 1.3 and 1.2, and listed on Table 1. So, we may assume that $n \ge 12$.

Suppose, by contradiction, that the upper bound for $\widehat{\lambda}_n(\xi)$ given by Theorem 1.1 does not hold for such n. Then, we may assume that

$$\lambda = \left(\frac{n}{2} + a\sqrt{n} + \frac{1}{3}\right)^{-1}$$

where $a = (1 - \log(2))/2$. Define

$$\ell = \left\lfloor \frac{n}{2} - \frac{\log(2)}{2}\sqrt{n} + 1 \right\rfloor, \quad k = n - 2\ell, \quad \vartheta = \frac{\ell\lambda}{1-\lambda} \quad \text{and} \quad \eta = \ell - 1 + \sum_{m=0}^{k+1} \vartheta^m.$$

Since $n \ge 9$, we find that $1 \le \ell < n/2$ and so $k \ge 1$. We will show further that

$$\frac{1}{2} \le \vartheta^k < 1 \quad \text{and} \quad \eta > \lambda^{-1}. \tag{10.3}$$

So this choice of parameters fulfills all the hypotheses of Proposition 10.1 but not its conclusion. This will conclude the proof, by contradiction.

A quick computer computation shows that (10.3) holds for $12 \le n < 900$. So, we may assume that $\sqrt{n} \ge 30$. This assumption will simplify the estimates below.

By choice of ℓ, there is some $t \in (0, 2]$ such that

$$\ell = (n - \log(2)\sqrt{n} + t)/2\,, \quad \text{and then} \quad k = \log(2)\sqrt{n} - t.$$

Using the actual value of λ, this gives

$$\vartheta = \frac{n-k}{2\big(\lambda^{-1} - 1\big)} = \frac{n - \log(2)\sqrt{n} + t}{n + 2a\sqrt{n} - 4/3}.$$

Define

$$\epsilon = 1 - \vartheta = \frac{\sqrt{n} - (4/3 + t)}{n + 2a\sqrt{n} - 4/3}. \tag{10.4}$$

As $n \ge 12$, we have $0 < \epsilon < 1$, thus $0 < \vartheta < 1$, and so $\vartheta^k < 1$. We set

$$r = -\frac{\log(2)}{\log(1-\epsilon)}$$

so that $\vartheta^r = (1-\epsilon)^r = 1/2$. Since $0 < \epsilon < 1$, we find

$$\frac{\epsilon}{1 - \epsilon/2} = \sum_{i=1}^{\infty} \frac{\epsilon^i}{2^{i-1}} \le \sum_{i=1}^{\infty} \frac{\epsilon^i}{i} = -\log(1-\epsilon) \le \sum_{i=1}^{\infty} \epsilon^i = \frac{\epsilon}{1-\epsilon},$$

thus, by definition of r,

$$\log(2)\left(\frac{1}{\epsilon} - 1\right) \le r \le \log(2)\left(\frac{1}{\epsilon} - \frac{1}{2}\right). \tag{10.5}$$

Moreover, (10.4) yields

$$\frac{1}{\epsilon} = \sqrt{n} + b + \frac{c}{\sqrt{n} - (4/3 + t)}$$

with $b = 2a + 4/3 + t$ and $c = b(4/3 + t) - 4/3$. As $c > 0$, this implies that

$$\frac{1}{\epsilon} \geq \sqrt{n} + b + \frac{c}{\sqrt{n}} \geq \sqrt{n} + 1.64, \tag{10.6}$$

so $r > \log(2)\sqrt{n} \geq k$, and thus $\vartheta^k \geq \vartheta^r = 1/2$. This proves the first condition in (10.3).

To verify the second condition $\eta > \lambda^{-1}$, we first note that

$$\frac{1}{\epsilon} \leq \sqrt{n} + b + \frac{c}{30 - (4/3 + 2)} \leq \sqrt{n} + 4.05, \tag{10.7}$$

using the hypothesis $\sqrt{n} \geq 30$ and $t \leq 2$. We set further

$$\delta = k + 2 - r \quad \text{and} \quad E = \frac{1 - \vartheta^\delta}{1 - \vartheta} - \delta.$$

As $\vartheta^r = 1/2$, we find

$$\begin{aligned} \eta = \ell - 1 + \frac{1 - \vartheta^{k+2}}{1 - \vartheta} &= \ell - 1 + \frac{1 - \vartheta^r}{1 - \vartheta} + \vartheta^r \frac{1 - \vartheta^\delta}{1 - \vartheta} \\ &= \frac{n - k}{2} - 1 + \frac{1}{2\epsilon} + \frac{1}{2}(E + \delta) = \frac{n}{2} - \frac{r}{2} + \frac{1}{2\epsilon} + \frac{E}{2}. \end{aligned}$$

Using the upper bound for r given by (10.5) and then the lower bound for $1/\epsilon$ given by (10.6), we deduce that

$$\begin{aligned} \eta - \lambda^{-1} &\geq \frac{n}{2} + \frac{a}{\epsilon} + \frac{\log(2)}{4} + \frac{E}{2} - \left(\frac{n}{2} + a\sqrt{n} + \frac{1}{3}\right) \\ &\geq 1.64\, a + \frac{\log(2)}{4} - \frac{1}{3} + \frac{E}{2} \\ &\geq 0.09 + \frac{E}{2}. \end{aligned}$$

So it remains to show that $E > -0.18$.

Lagrange remainder theorem gives

$$\vartheta^\delta = (1 - \epsilon)^\delta = 1 - \delta\epsilon + \frac{\delta(\delta - 1)}{2}(1 - \epsilon')^{\delta - 2}\epsilon^2$$

for some real number ϵ' with $0 < \epsilon' < \epsilon$, and thus

$$E = \frac{1 - \vartheta^\delta}{\epsilon} - \delta = -\frac{\delta(\delta - 1)}{2}(1 - \epsilon')^{\delta - 2}\epsilon.$$

As $k < r$, we have $\delta < 2$. Moreover, the upper bound for r given by (10.5) together with that of $1/\epsilon$ given by (10.7) yields

$$\delta \geq k + 2 - \log(2)\left(\frac{1}{\epsilon} - \frac{1}{2}\right) \geq -2.47.$$

Since by (10.6) we have $\epsilon < 1/\sqrt{n} \leq 1/30$, we conclude that

$$|E| \leq \frac{2.47 \times 3.47}{2}\left(1 - \frac{1}{30}\right)^{-4.47}\frac{1}{30} \leq 0.17.$$

□

11. A new construction

In this section, we introduce a new construction which in some cases yields $\lambda_k(\xi) > 1$ for an integer $k \geq 1$. We will use it in the proof of Theorems 1.2 and 1.3 in combination with the following results of Schleischitz [21, Theorems 1.6 and 1.12].

Theorem 11.1 (Schleischitz, 2016). *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$. For each integer $n \geq 1$, we have*

$$\widehat{\lambda}_n(\xi) \leq \max\{1/n, 1/\lambda_1(\xi)\}.$$

Moreover, if $\lambda_k(\xi) > 1$ for some integer $k \geq 1$, then $\lambda_1(\xi) = k - 1 + k\lambda_k(\xi)$.

In fact, we will simply need the following weaker consequence.

Corollary 11.2. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and let $\lambda \in (1/n, 1)$. Suppose that, for some integer $k \geq 1$, there exist non-zero points $C \in \mathbb{Z}^{k+1}$ for which $L_\xi(C)$ is arbitrarily small while the products $\|C\| L_\xi(C)^\lambda$ remain bounded from above. Then, we have $\widehat{\lambda}_n(\xi) \leq \lambda$.*

Proof. The hypothesis implies that $\lambda_k(\xi) \geq 1/\lambda > 1$. By the above theorem, we conclude that $\lambda_1(\xi) \geq 1/\lambda$ and so $\widehat{\lambda}_n(\xi) \leq \max\{1/n, \lambda\} = \lambda$. □

For each positive integer k and each non-zero subspace V of $\mathbb{R}^{k+1}$ defined over $\mathbb{Q}$, it is natural to define

$$L_\xi(V) = \|\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_s \wedge \Xi_k\|$$

where $\{\mathbf{z}_1, \ldots, \mathbf{z}_s\}$ is a basis of $V \cap \mathbb{Z}^{k+1}$ over $\mathbb{Z}$, and where $\Xi_k = (1, \xi, \ldots, \xi^k)$. This is independent of the choice of the basis, like for the height $H(V) = \|\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_s\|$ of V. In particular, if $\mathbf{z}$ is a primitive point of $\mathbb{Z}^{k+1}$, we find

$$L_\xi(\langle \mathbf{z} \rangle) = \|\mathbf{z} \wedge \Xi_k\| \asymp L_\xi(\mathbf{z})$$

with an implied constant that depends only on k and ξ (see §2). In general, if $\{\mathbf{y}_1, \ldots, \mathbf{y}_s\}$ is a maximal linearly independent subset of $V \cap \mathbb{Z}^{k+1}$, then arguing as in the proof of Lemma 2.1, we find

$$L_\xi(V) \leq \|\mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_s \wedge \Xi_k\| \ll L_\xi(\mathbf{y}_1) \cdots L_\xi(\mathbf{y}_s),$$

with an implied constant of the same nature. We can now present our construction.

Proposition 11.3. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$, let $k, \ell \geq 1$ be integers, and let $n = k + \ell$. Suppose that V is a subspace of $\mathbb{R}^{k+1}$ of dimension k, and that $\mathbf{x} \in \mathbb{Z}^{n+1}$ satisfies $\mathcal{U}^\ell(\mathbf{x}) \not\subseteq V$. Finally, let $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ be a basis of $V \cap \mathbb{Z}^{k+1}$. Then the point*

$$C = \big(\det(\mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{x}^{(0,\ell)}), \ldots, \det(\mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{x}^{(\ell,\ell)})\big) \in \mathbb{Z}^{\ell+1}$$

is non-zero. It satisfies

$$\|C\| \ll \|\mathbf{x}\| L_\xi(V) + H(V) L_\xi(\mathbf{x}) \quad \text{and} \quad L_\xi(C) \ll H(V) L_\xi(\mathbf{x})$$

with implied constants that depend only on k and ξ.

We will write $C(V,\mathbf{x})$ to denote this point C, although it is determined by V and $\mathbf{x}$ only up to multiplication by ± 1. In practice, this is no problem since this ambiguity does not affect the quantities $\|C\|$ and $L_\xi(C)$.

Proof. Write $\mathbf{x} = (x_0,\dots,x_n)$ and $C = (C_0,\dots,C_\ell)$. Then we have $\mathbf{x} = x_0\Xi_n + \Delta$ with $\|\Delta\| \asymp L_\xi(\mathbf{x})$ and for each $j = 0,\dots,\ell$ we find

$$C_j = x_0\xi^j \det(\mathbf{z}_1,\dots,\mathbf{z}_k,\Xi_k) + \det(\mathbf{z}_1,\dots,\mathbf{z}_k,\Delta^{(j,\ell)}).$$

The estimates for $\|C\|$ and $L_\xi(C) = \max_{1\le j\le k} |C_j - C_0\xi^j|$ follow. □

12. Small odd degree

This section is devoted to the proof of Theorem 1.2. So, we suppose that

$$n = 2m+1 \ge 5$$

is an odd integer. We argue by contradiction, assuming that $\widehat{\lambda}_n(\xi) > \alpha$ where $\alpha = \alpha_m$ is the unique positive root of

$$P_m(x) = 1-(m+1)x - mx^2.$$

Then $\mathcal{P}(1,m)$ holds by Corollary 7.5, and so $\mathcal{P}(2,m-1)$ holds as well by Proposition 5.6. Thus, for each large enough $i \in I$, the vector space

$$V_i = \mathcal{U}^{m-1}(\mathbf{x}_{i-1},\mathbf{x}_i,\mathbf{x}_{i+1}) \subseteq \mathbb{R}^{m+3}$$

has dimension at least $m+2$ (see the definitions and remarks in Section 5).

Lemma 12.1. *With the above hypotheses, we have*

$$X_{j+1}^{\vartheta} \ll X_{i+1} \quad \textit{and} \quad X_{i+1}^{\vartheta} \ll X_i \quad \textit{where} \quad \vartheta = \frac{m\alpha}{1-\alpha} = \frac{1}{1+\alpha}, \tag{12.1}$$

for each pair of consecutive elements $i<j$ of I. If i is large enough, then $\dim(V_i) = m+2$,

$$H(V_i) \ll X_{j+1}^{-(m-1)\alpha} X_{i+1}^{1-\alpha} X_i^{-\alpha} \quad \textit{and} \quad L_\xi(V_i) \ll X_{j+1}^{-m\alpha} X_{i+1}^{-\alpha} X_i^{-\alpha}. \tag{12.2}$$

Proof. Since $\widehat{\lambda}_n(\xi) > \alpha$, we can choose $\epsilon > 0$ such that $\alpha + \epsilon < \widehat{\lambda}_n(\xi)$. Then the results of the preceding sections apply with $\lambda = \alpha+\epsilon$. In particular, for each pair of consecutive elements $i<j$ of I, Lemma 7.2 gives

$$H(\mathcal{U}^m(\mathbf{x}_i,\mathbf{x}_{i+1})) \ll X_{j+1}^{-m\alpha} X_{i+1}^{1-\alpha} \quad \text{and} \quad H(V_i) \ll X_{j+1}^{-(m-1)\alpha} X_{i+1}^{1-\alpha-\epsilon} X_i^{-e(i)\alpha}$$

where $e(i) = \dim(V_i) - (m+1)$, because both $\mathcal{P}(1,m)$ and $\mathcal{P}(1,m-1)$ hold. Since we have $H(\mathcal{U}^m(\mathbf{x}_i,\mathbf{x}_{i+1})) \ge 1$, the first estimate yields $X_{j+1}^{\vartheta} \ll X_{i+1}$ with ϑ as in (12.1) (this also follows from Corollary 9.4). So, if i is large enough to admit a predecessor $h<i$ in I, we also have $X_{i+1}^{\vartheta} \ll X_{h+1} \le X_i$, thus $X_{i+1} \ll X_i^{1+\alpha}$. This proves (12.1). If furthermore $V_i = \mathbb{R}^{m+3}$, then $e(i)=2$ and using $X_{j+1} \ge X_{i+1}$, we obtain

$$1 = H(V_i) \ll X_{i+1}^{1-m\alpha-\epsilon} X_i^{-2\alpha} \ll X_i^{(1+\alpha)(1-m\alpha-\epsilon)-2\alpha} = X_i^{-(1+\alpha)\epsilon},$$

which forces i to be bounded. So, if i is large enough, we have $\dim(V_i) = m+2$, thus $e(i) = 1$ and the estimate for $H(V_i)$ in (12.2) follows. Finally, the proof of Lemma 7.2 shows that V_i admits a basis of the form

$$\{\mathbf{x}_j^{(0,m-1)}, \dots, \mathbf{x}_j^{(m-1,m-1)}, \mathbf{x}_{j-1}^{(p,m-1)}, \mathbf{x}_h^{(q,m-1)}\}$$

for some $p, q \in \{0, 1, \dots, m-1\}$ and some $h \in \{i-1, j-1\}$. So the considerations of the preceding section provide $L_\xi(V_i) \ll L_j^m L_{j-1} L_{i-1} \le X_{j+1}^{-m\alpha} X_{i+1}^{-\alpha} X_i^{-\alpha}$. □

It is now an easy matter to complete the proof of Theorem 1.2. By the preceding lemma, there are arbitrarily large pairs of successive elements $i < j$ of I for which $\dim(V_i) = m+2$ and $V_j \not\subseteq V_i$. The latter condition means that $\mathcal{U}^{m-1}(\mathbf{x}_{j+1}) \not\subseteq V_i$. So, for these pairs, Proposition 11.3 shows that the point

$$C_i = C(V_i, \mathbf{x}_{j+1}) \in \mathbb{Z}^m$$

is non-zero. Using the estimates of the preceding lemma, it also gives

$$\begin{aligned} L_\xi(C_i) &\ll H(V_i) L_{j+1} \ll X_{j+1}^{-m\alpha} X_{i+1}^{1-\alpha} X_i^{-\alpha}, \\ \|C_i\| &\ll X_{j+1} L_\xi(V_i) + H(V_i) L_{j+1} \ll X_{j+1}^{1-m\alpha} X_{i+1}^{-\alpha} X_i^{-\alpha}. \end{aligned}$$

Using (12.1), we find that

$$\begin{aligned} \|C_i\| L_\xi(C_i)^\alpha \ll X_{j+1}^{1-m\alpha-m\alpha^2} X_{i+1}^{-\alpha^2} X_i^{-\alpha-\alpha^2} &= X_{j+1}^{\alpha} X_{i+1}^{-\alpha^2} X_i^{-\alpha/\vartheta} \\ &\ll X_{j+1}^{\alpha} X_{i+1}^{-\alpha^2-\alpha} = X_{j+1}^{\alpha} X_{i+1}^{-\alpha/\vartheta} \ll 1 \end{aligned}$$

is bounded from above, and that

$$L_\xi(C_i) \ll X_{j+1}^{-m\alpha} X_{i+1}^{1-\alpha} X_i^{-\alpha} \ll X_{j+1}^{-m\alpha} X_{i+1}^{1-\alpha-\alpha\vartheta} \le X_{j+1}^{-m\alpha+1-\alpha-\alpha\vartheta} = X_{j+1}^{-\alpha^2\vartheta}$$

tends to 0 as i goes to infinity. By Corollary 11.2, this implies that $\widehat{\lambda}_n(\xi) \le \alpha$.

13. Small even degree

We conclude, in this section, with the proof of Theorem 1.3. So, we assume that

$$n = 2m \ge 4$$

is an even integer. For the proof, define $\beta = \beta_m$ to be the single positive root of

$$Q_m(x) = \begin{cases} 1 - mx - mx^2 - m(m-1)x^3 & \text{if } m \ge 3, \\ 1 - 3x + x^2 - 2x^3 - 2x^4 & \text{if } m = 2, \end{cases}$$

as in the statement of the theorem, then write

$$\gamma = \begin{cases} \dfrac{m+4}{m^2+6m+2} & \text{if } m \ge 4, \\ 8/33 & \text{if } m = 3, \\ 1/3 & \text{if } m = 2, \end{cases}$$

and define δ to be the single positive root of

$$R_m(x) = 1 - (m+1)x - (m-1)x^2.$$

It is a simple matter to check that

$$1/(m+2) < \delta < \gamma < \beta.$$

We will prove that $\widehat{\lambda}_n(\xi) \le \beta$ through the following chains of implications

$$\begin{aligned} \widehat{\lambda}_n(\xi) > \delta \ &\Longrightarrow\ \mathcal{P}(1, m-1) \text{ holds} &&\Longrightarrow\ \mathcal{P}(2, m-2) \text{ holds} \\ \widehat{\lambda}_n(\xi) > \gamma \ &\Longrightarrow\ \mathcal{P}(2, m-1) \text{ does not hold} &&\Longrightarrow\ \widehat{\lambda}_n(\xi) \le \beta. \end{aligned}$$

Recall that λ represents a fixed real number with $0 < \lambda < \widehat{\lambda}_n(\xi)$.

Lemma 13.1. *Suppose that $\widehat{\lambda}_n(\xi) > \delta$. Then both $\mathcal{P}(1, m-1)$ and $\mathcal{P}(2, m-2)$ hold. Moreover, we have $X_{i+1}^{m\lambda} \ll X_i$ for each $i \ge 0$.*

Proof. For the choice of $\ell = m-1$, we have $1 \le \ell < n/2$ and $R_m(x) = 1 - (n-\ell)x - \ell x^2$. Thus, by Corollary 7.5, our hypothesis implies $\mathcal{P}(1, m-1)$ which in turn implies $\mathcal{P}(2, m-2)$, by the general Proposition 5.6. Since $\delta \ge 1/(m+2) \ge 1/(2m)$, the growth estimate follows from Lemma 7.3. □

Lemma 13.2. *Suppose that $\mathcal{P}(2, m-1)$ holds and choose $\vartheta \in (0, 1]$ such that*

$$\frac{m-1}{\vartheta} + \vartheta > \frac{1}{\lambda} - 1. \tag{13.1}$$

Then we have $X_{j+1}^{\vartheta} \le X_{i+1}$ for any large enough pair of consecutive elements $i < j$ of I.

Proof. Property $\mathcal{P}(2, m-1)$ means that $\mathcal{U}^{m-1}(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}) = \mathbb{R}^{m+2}$ for each large enough $i \in I$. By definition, it also implies property $\mathcal{P}(1, m-1)$. Thus, for all but finitely many triples $h < i < j$ of consecutive elements of I, we obtain, by Lemma 7.2,

$$1 = H(\mathcal{U}^{m-1}(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1})) \ll X_{j+1}^{-(m-1)\lambda} X_{i+1}^{1-\lambda} X_{h+1}^{-\lambda}$$

using the crude estimate $X_i \ge X_{h+1}$. Taking logarithms, this gives

$$(m-1)\frac{\log(X_{j+1})}{\log(X_{i+1})} + \frac{\log(X_{h+1})}{\log(X_{i+1})} \le \frac{1-\lambda}{\lambda} + \mathcal{O}\Big(\frac{1}{\log(X_{i+1})}\Big).$$

Let $\mu \in [0, 1]$ denote the inferior limit of the ratio $\log(X_{i+1})/\log(X_{j+1})$ as (i, j) runs through the pairs of consecutive elements $i < j$ of I. Then, there is a sequence of triples (h, i, j), with i going to infinity and $h < i < j$ consecutive in I, for which the ratio $\log(X_{i+1})/\log(X_{j+1})$ converges to μ. Over that sequence, the inferior limit of $\log(X_{h+1})/\log(X_{i+1})$ is at least μ, and so the above inequality implies that $\mu > 0$ and

$$\frac{m-1}{\mu} + \mu \le \frac{1}{\lambda} - 1.$$

As $m \ge 2$, the expression $(m-1)/x + x$ is a strictly decreasing function of x on $(0, 1]$. We conclude that our choice of ϑ satisfies $0 < \vartheta < \mu$, and so $\vartheta \le \log(X_{i+1})/\log(X_{j+1})$ for any pair of consecutive elements $i < j$ of I with i large enough. □

Lemma 13.3. *Suppose that* $\widehat{\lambda}_n(\xi) > \gamma$. *Then* $\mathcal{P}(2, m-1)$ *does not hold.*

Proof. Suppose on the contrary that $\mathcal{P}(2, m-1)$ holds. By the hypothesis, we may assume that $\lambda > \gamma$. A short computation shows that $(m-1)/\vartheta + \vartheta \ge 1/\gamma - 1$ for

$$\vartheta = \begin{cases} 1 & \text{if } m = 2, \\ 15/17 & \text{if } m = 3, \\ (m+4)/(m+5) & \text{if } m \ge 4. \end{cases}$$

Thus this choice of ϑ fulfills the main condition (13.1) of Lemma 13.2, and so we have $X_{j+1}^{\vartheta} \le X_{i+1}$ for each large enough pair of consecutive elements $i < j$ of I. For $m = 2$, this becomes $X_{j+1} \le X_{i+1}$ which is already a contradiction because the sequence $(X_i)_{i\ge 0}$ is strictly increasing. Thus, we may assume that $m \ge 3$.

Since $\mathcal{P}(2, m-1)$ holds, Proposition 5.6 implies that $\mathcal{P}(3, m-2)$ holds as well. So, for each large enough i, the subspace $V_i = \mathcal{U}^{m-2}(A_3(i))$ of $\mathbb{R}^{m+3}$ has dimension at least $m+2$ and Proposition 8.2 gives

$$H(V_i) \ll Y_2(i)^{1-(m-1)\lambda} Y_1(i)^{-\lambda} Y_0(i)^{-(1+e(i))\lambda} \tag{13.2}$$

where $e(i) = \dim(V_i) - m - 2 \in \{0, 1\}$. If $V_i \ne \mathbb{R}^{m+3}$ for infinitely many i, then Lemma 6.1 provides arbitrarily large $i \in I$ for which $\mathcal{U}^{m-2}(\mathbf{x}_{i-1}) \not\subseteq V_i$. For those i, we have $e(i) = 0$ and Lemma 6.2 gives $1 \ll H(V_i) L_{i-1}$. Then, we obtain

$$1 \ll Y_2(i)^{1-(m-1)\lambda} Y_1(i)^{-\lambda} Y_0(i)^{-\lambda} X_i^{-\lambda}. \tag{13.3}$$

Otherwise, we have $e(i) = 1$ for all sufficiently large $i \in I$ and the above estimate follows directly from (13.2) since $1 \le H(V_i)$ and $Y_0(i) = X_{i+1} > X_i$. Thus (13.3) holds for infinitely many $i \in I$. Viewing such i as part of a triple of consecutive elements $h < i < j$ of I, we have $Y_1(i) = X_{j+1}$, $Y_0(i) = X_{i+1}$ and $X_i \ge X_{h+1}$, thus

$$Y_1(i)^{\vartheta} \ll Y_0(i) \quad \text{and} \quad Y_0(i)^{\vartheta} \ll X_i \tag{13.4}$$

by our initial observation at the beginning of the proof. So, we deduce that

$$Y_2(i)^{1-(m-1)\lambda} \gg Y_1(i)^{\lambda(1+\vartheta+\vartheta^2)} \gg Y_1(i)^{3\lambda\vartheta} \tag{13.5}$$

for arbitrarily large $i \in I$. Since $m \ge 3$, we may also apply Corollary 9.3 with $j = 2$ and $\ell = m-1$. Using (13.4), this gives

$$Y_2(i)^{(m-2)\lambda} \ll Y_1(i)^{1-2\lambda} Y_0(i)^{-\lambda} \ll Y_1(i)^{1-2\lambda-\lambda\vartheta}$$

for each $i \in I$. Substituting this upper bound for $Y_2(i)$ into (13.5) and then comparing powers of $Y_1(i)$, we conclude that

$$3(m-2)\lambda^2\vartheta \le (1-(m-1)\lambda)(1-2\lambda-\lambda\vartheta),$$

and thus $3(m-2)\vartheta \le (1/\gamma - (m-1))(1/\gamma - 2 - \vartheta)$, as λ can be taken arbitrarily close to γ. However, this inequality is false for the actual values of γ and ϑ. This contradiction shows that $\mathcal{P}(2, m-1)$ does not hold. □

Proof of Theorem 1.3. We may assume that $\gamma < \lambda < \widehat{\lambda}_n(\xi)$. Then $\mathcal{P}(1, m-1)$ and $\mathcal{P}(2, m-2)$ hold while $\mathcal{P}(2, m-1)$ does not hold, by Lemmas 13.1 and 13.3. In particular, Proposition 6.3 applies with $j = 2$ and $\ell = m-1$, and so there are infinitely many integers $i \ge 1$ for which

$$\dim \mathcal{U}^{m-1}(A_2(i)) = m+1 \quad \text{and} \quad \mathcal{U}^{m-1}(\mathbf{x}_{i-1}) \not\subseteq \mathcal{U}^{m-1}(A_2(i)) \subsetneq \mathbb{R}^{m+2}. \tag{13.6}$$

For those i, Lemma 6.2 and Proposition 6.3 further give

$$1 \ll H(\mathcal{U}^{m-1}(A_2(i)))L_{i-1} \quad \text{and} \quad 1 \ll H(\mathcal{U}^{m-2}(A_2(i)))L_{i-1}^2.$$

Any such i belongs to I and, upon denoting by j its successor in I, Proposition 8.2 gives

$$H(\mathcal{U}^{m-2}(A_2(i))) \ll Y_1(i)^{1-(m-1)\lambda} Y_0(i)^{-\lambda} = X_{j+1}^{1-(m-1)\lambda} X_{i+1}^{-\lambda}.$$

By $\mathcal{P}(1, m-1)$, we also have $\dim \mathcal{U}^{m-1}(A_1(i)) \ge m+1$ if i is large enough. Comparing with (13.6), this implies that $\mathcal{U}^{m-1}(A_2(i)) = \mathcal{U}^{m-1}(A_1(i))$ and so Lemma 7.2 gives

$$H(\mathcal{U}^{m-1}(A_2(i))) = H(\mathcal{U}^{m-1}(A_1(i))) \ll X_{j+1}^{-(m-1)\lambda} X_{i+1}^{1-\lambda}.$$

By Lemma 13.1, we also have $L_{i-1} \ll X_i^{-\lambda} \ll X_{i+1}^{-m\lambda^2}$. Combining all the above inequalities, we obtain

$$1 \ll X_{j+1}^{-(m-1)\lambda} X_{i+1}^{1-\lambda-m\lambda^2} \quad \text{and} \quad 1 \ll X_{j+1}^{1-(m-1)\lambda} X_{i+1}^{-\lambda-2m\lambda^2}. \tag{13.7}$$

As we can take i arbitrarily large, this in turn implies that

$$0 \le (1-(m-1)\lambda)(1-\lambda-m\lambda^2) - (m-1)\lambda(\lambda+2m\lambda^2).$$

If $m \ge 3$, the right hand side of this inequality simplifies to $Q_m(\lambda)$. So, in that case, we obtain $\lambda \le \beta$, thus $\widehat{\lambda}_n(\xi) \le \beta = \beta_m$ as needed.

For the case $m = 2$, we look more closely at the vector spaces

$$V_i = \mathcal{U}^1(A_1(i-1)) = \mathcal{U}^1(\mathbf{x}_{i-1}, \mathbf{x}_i) \subseteq \mathbb{R}^4$$

for each integer $i \ge 1$. Since $\mathcal{P}(1,1)$ holds, we have $\dim \mathcal{U}^1(\mathbf{x}_i) = 2$ and $\dim(V_i) \ge 3$ for each large enough i. When $V_i = \mathbb{R}^4$, we find $1 = H(V_i) \ll X_i L_{i-1}^3 \ll X_i^{1-3\lambda}$ since V_i is generated by points $\mathbf{y} \in \mathbb{Z}^4$ with $\|\mathbf{y}\| \le X_i$ and $L_\xi(\mathbf{y}) \ll L_{i-1}$. As $\lambda > \gamma = 1/3$, we conclude that both $\dim \mathcal{U}^1(\mathbf{x}_i) = 2$ and $\dim(V_i) = 3$ for each large enough i, say for $i \ge i_0$. Then, V_i admits a basis of the form $\{\mathbf{x}_{i-1}^{(p,1)}, \mathbf{x}_i^{(0,1)}, \mathbf{x}_i^{(1,1)}\}$ for some $p \in \{0,1\}$, thus

$$H(V_i) \ll X_i L_i L_{i-1} \ll X_{i+1}^{-\lambda} X_i^{1-\lambda} \quad \text{and} \quad L_\xi(V_i) \ll L_i^2 L_{i-1} \ll X_{i+1}^{-2\lambda} X_i^{-\lambda}.$$

For each $i \ge i_0$ for which (13.6) holds, we have $\mathcal{U}^1(\mathbf{x}_{i-1}) \not\subseteq V_{i+1}$, thus $V_i \cap V_{i+1} = \mathcal{U}^1(\mathbf{x}_i)$ and so $\mathcal{U}^1(\mathbf{x}_{i+1}) \not\subseteq V_i$. Then, by Proposition 11.3, the point $C_i = C(V_i, \mathbf{x}_{i+1}) \in \mathbb{Z}^2$ is non-zero with

$$\|C_i\| \ll H(V_i)L_{i+1} + X_{i+1} L_\xi(V_i) \ll X_{i+1}^{1-2\lambda} X_i^{-\lambda}.$$

Moreover, by Lemma 5.1, the pair $\{\mathbf{x}_i, \mathbf{x}_{i+1}\}$ is a basis of $A_1(i) \cap \mathbb{Z}^5$ over $\mathbb{Z}$. So, letting j denote the successor of i in I, we may write $\mathbf{x}_j = a\mathbf{x}_i + b\mathbf{x}_{i+1}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$.

Since $C(V_i, \mathbf{x})$ is linear in $\mathbf{x}$ (for a fixed basis of V_i) and since $C(V_i, \mathbf{x}_i) = 0$, we find that $C(V_i, \mathbf{x}_j) = bC_i$. Thus, by Proposition 11.3, we obtain

$$L_\xi(C_i) \le L_\xi(C(V_i, \mathbf{x}_j)) \ll H(V_i)L_j \ll X_{j+1}^{-\lambda} X_{i+1}^{-\lambda} X_i^{1-\lambda}.$$

In particular, $L_\xi(C_i) \ll X_i^{1-3\lambda}$ converges to 0 as $i \to \infty$, since $\lambda > 1/3$. Thus, by Corollary 11.2, the product $\|C_i\| L_\xi(C_i)^\lambda$ tends to infinity with i. So we have

$$1 \ll \|C_i\| L_\xi(C_i)^\lambda \ll X_{j+1}^{-\lambda^2} X_{i+1}^{1-2\lambda-\lambda^2} X_i^{-\lambda^2}.$$

Using the estimate $X_{i+1}^{2\lambda} \ll X_i$ from Lemma 13.1, we conclude that

$$X_{j+1}^{\lambda^2} \ll X_{i+1}^{1-2\lambda-\lambda^2-2\lambda^3} \tag{13.8}$$

for each pair of consecutive elements $i < j$ of I with $i \ge i_0$, for which (13.6) holds. For these, the two estimates (13.7) also apply. In particular, the second one yields

$$X_{i+1}^{\lambda+4\lambda^2} \ll X_{j+1}^{1-\lambda}.$$

Combining this with (13.8), we conclude that $\lambda^2(\lambda + 4\lambda^2) \le (1-\lambda)(1-2\lambda-\lambda^2-2\lambda^3)$, which simplifies to $Q_2(\lambda) \ge 0$. This gives $\lambda \le \beta$ and thus $\widehat{\lambda}_4(\xi) \le \beta = \beta_2$.

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