On the linear independence for p-adic polygamma values

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Abstract

In this article, we present a new linear independence criterion for values of the p-adic polygamma functions defined by J. Diamond. As an application, we obtain the linear independence of some families of values of the p-adic Hurwitz zeta function $\zeta_p(s,x)$ at distinct shifts x. This improves and extends a previous result due to P. Bel [5], as well as irrationality results established by F. Beukers [7]. Our proof is based on a novel and explicit construction of Padé-type approximants of the second kind of Diamond's p-adic polygamma functions. This construction is established by using a difference analogue of the Rodrigues formula for orthogonal polynomials.

Key words and phrases: p-adic values, p-adic polygamma functions, log gamma function of Diamond, p-adic Hurwitz zeta function, irrationality, linear independence, Padé approximation.

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1 Introduction

1.1 State of the art

A classical problem in Diophantine approximation is the study of the irrationality or linear independence of values of L-functions at positive integers. A famous example is the case of the Riemann zeta function. In their seminal 2001 papers [3, 38], K. Ball and T. Rivoal established that, given an odd integer $a \geq 3$, the dimension $\delta(a)$ of the $\mathbb{Q}$-vector space spanned by $1, \zeta(3), \zeta(5), \dots, \zeta(a)$ satisfies

$$\delta(a) \geq \frac{1}{3}\log a,$$

where $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is the Riemann zeta function. Their work has inspired many others, see for example S. Fischler, Sprang and W. Zudilin [19], L. Lai and P. Yu [28], and Fischler [18] for irrationality and linear independence results of odd values of the Riemann zeta function. In this article, we are interested in proving similar properties for p-adic L-functions, where p denotes a fixed prime number. We still have few answers to this kind of questions. In 2005, F. Calegari [9] used the theory of p-adic modular forms to establish the irrationality of $\zeta_p(3)$, for $p = 2, 3$, and $L_2(2, \chi)$, where χ is the Dirichlet character of conductor 4. Subsequently, F. Beukers [7] provided an alternative proof of these results. Using classical continued fractions discovered by T. J. Stieltjes, he also proved the irrationality for some values of the p-adic Hurwitz zeta function $\zeta_p(s,x)$ at $s = 2, 3$ (see Definition 2.3 of Section 2 for the precise definition of $\zeta_p(s,x)$). In [5], P. Bel adapted the approach of [3] to obtain similar properties for certain p-adic functions. This was later generalized by M. Hirose, N. Sato and the first author [22]. The following linear independence criterion is a consequence of the proof of [5, Théorème 3.2].

Theorem 1.1 (Bel, 2010). *Let $m \geq 1$ be an integer and p be a prime number such that*

$$\log p \geq (1 + \log 2)(m+1)^2. \tag{1}$$

Then, the $m+1$ elements of $\mathbb{Q}_p$:

$$1, \zeta_p(2, p^{-1}), \ldots, \zeta_p(m+1, p^{-1})$$

are linearly independent over $\mathbb{Q}$.

Bel proved Theorem 1.1 building upon a construction of Padé first kind approximations due to T. Rivoal (*confer* [39]), which is derived from the asymptotic expansion of the Hurwitz zeta function. In [6] Bel also proved the irrationality of $\zeta_p(4, x)$ for $x = 1/p$ with $p \geq 19$. Lower bounds for the dimension of the vector space spanned by p-adic Hurwitz zeta values and Kubota-Leopoldt p-adic L-values can be found in the recent papers of J. Sprang [44], also see L. Lai [30], and Lai and Sprang [31]. Their results were obtained by constructing approximations of p-adic L-values, in a similar way to Ball-Rivoal for the (complexe) Riemann zeta function [3]. Before stating our main result, which requires some technical definitions, let us present two of its consequences.

THEOREM 1.2. *There exists an explicit constant $C \geq 1$ with the following properties. Let p be a prime number and m, r be positive integers. Suppose that*

$$r \log p \geq Cm\log(m+1). \tag{2}$$

Then the $m+1$ elements of $\mathbb{Q}_p$:

$$1, \zeta_p(2, p^{-r}), \ldots, \zeta_p(m+1, p^{-r})$$

are linearly independent over $\mathbb{Q}$.

For $r = 1$, we improve Theorem 1.1 by replacing the condition $\log p \gg m^2$ with the weaker condition $\log p \gg m \log m$. Note that for a fixed parameter m, condition (2) is automatically satisfied is p is large enough. A refined and explicit version of Theorem 1.2, namely Theorem 12.1, is given in Section 12. Also note that the special case $d = m = 1$ of the aforementioned result (see Section 12) was proved by Beukers in [7, Theorem 9.2]. The table below shows how the condition in Theorem 12.1 compares to that of Bel (B.) for $m \leq 8$. For $m \geq 3$, we just wrote a crude order of magnitude given by the conditions.

m	1	2	3	4	5	6	7	8
$p \geq$ (B.)	874	4148779	$6 \cdot 10^{11}$	$2 \cdot 10^{18}$	$3 \cdot 10^{26}$	10^{36}	10^{47}	$4 \cdot 10^{59}$
$p \geq$ (new)	5	144	$7 \cdot 10^{6}$	10^{9}	$3 \cdot 10^{11}$	$7 \cdot 10^{13}$	$2 \cdot 10^{16}$	$8 \cdot 10^{18}$

Figure 1: Comparison between our condition and that of P. Bel

Our main theorem also allows us to obtain the linear independence of values of the p-adic Hurwitz zeta function at distinct shifts. Theorem 12.2 of Section 11 is an explicit and refined version of the following result.

THEOREM 1.3. *There exists an explicit constant $C \geq 1$ with the following properties. Let p be a prime number and a, b, m, δ be positive integers with $\delta = a - 3(m+1)b > 0$. Assume that*

$$\left(\delta - \frac{3m+2}{p-1}\right) \log p \geq Cm\log(m+1). \tag{3}$$

Then the $2m+1$ elements of $\mathbb{Q}_p$:

$$1, \zeta_p(2, p^{-a}), \ldots, \zeta_p(m+1, p^{-a}), \zeta_p\big(2, p^{-a} + p^{-b}\big), \ldots, \zeta_p\big(m+1, p^{-a} + p^{-b}\big)$$

are linearly independent over $\mathbb{Q}$.

Remark 1.4. Similarly to condition (2) of Theorem 1.2, for fixed parameters a, b, m satisfying $\delta > 0$, condition (3) holds for all large enough p. More precisely, it is sufficient to have

$$p \geq \max\left\{\frac{2(3m+2)}{\delta} + 1, \exp\left(\frac{2Cm\log(m+1)}{\delta}\right)\right\}.$$

1.2 Main result

Let $\zeta(s,x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^s}$ denote the Hurwitz zeta function (where $s \in \mathbb{C}$ satisfies $\Re(s) > 1$ and $x \in \mathbb{R}_{>0}$). Consider $m \in \mathbb{N}$ with $m \geq 2$ and $x = a/b \in \mathbb{Q}$, where a and b are coprime positive integers. It is well known that $\zeta(m,x)$ is a period in the sense of Kontsevich and Zagier (see [27, Chapter 1]). Many fascinating and still widely open problems, such as Chowla-Milnor conjecture (see [21]), concern the linear independence over $\mathbb{Q}$ of values of $\zeta(m,x)$. In this article, we turn our attention to a p-adic analogue of this kind of questions. Values of Kubota-Leopoldt p-adic L-functions at positive integers can be expressed as linear combinations of values of p-adic polygamma functions at rational points, as shown by J. Diamond (*confer* [17, Theorem 3]). This motivates us to investigate in this paper the linear independence of p-adic polygamma values. In order to state our main result, namely Theorem 1.5 below, we need the following notation.

Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. We write $|\cdot|_p : \overline{\mathbb{Q}}_p \longrightarrow \mathbb{R}_{\geq 0}$ for the p-adic norm on $\overline{\mathbb{Q}}_p$ with the normalization $|p|_p = p^{-1}$. We denote by $G_p : \mathbb{Q}_p \setminus \mathbb{Z}_p \to \mathbb{Q}_p$ the log gamma function of Diamond, and by $\omega : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ the Teichmüller character, whose precise definitions are recalled in Section 2. The function G_p is a p-adic analog of the classical $\log\Gamma$-function and satisfies $G_p(x+1) = G_p(x) + \log_p(x)$, where $\log_p$ stands for the Iwasawa p-adic logarithm. The function G_p is locally analytic on $\mathbb{C}_p \setminus \mathbb{Z}_p$ and admits the expansion

$$G_p(z) = \left(z - \frac{1}{2}\right)\log_p(z) - z + \sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)} \cdot \frac{1}{z^k},$$

valid for each $z \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, where B_k denotes the k-th Bernoulli number. For any integer $s \geq 0$, the $(s+1)$-th derivative $G_p^{(s+1)}$ of G_p is called the *s-th Diamond's p-adic polygamma function*. We set $q_p = p$ if $p \geq 3$ and $q_p = 4$ if $p = 2$. The p-adic Hurwitz zeta function $\zeta_p(s,x)$ satisfies the classical identity

$$G_p^{(s)}(x) = (-1)^s (s-1)!\,\omega(x)^{1-s} \zeta_p(s,x) \tag{4}$$

for each $x \in \mathbb{Q}_p$ with $|x|_p \geq q_p$ and each integer $s \geq 2$ (see Eq. (16)). We are now ready to state the main results of our paper. To the best of our knowledge, this is the first p-adic linearly independence criterion involving distinct shifts $x + \alpha_i$.

Theorem 1.5. *There exists an absolute explicit constant $C \geq 1$ with the following properties. Let d, m be positive integers and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Q}^d$ with $\alpha_1 = 0$ and*

$$\alpha_i - \alpha_j \notin \mathbb{Z} \quad \text{for any } i \neq j. \tag{5}$$

Let $x \in \mathbb{Q}$. Suppose that $x, \alpha_2, \ldots, \alpha_d$ can be written as reduced fractions whose denominator is a power of p, and that

$$\log|x|_p \geq Cdm\Big(\log(dm+1) + \log\big(\max\{1, |\alpha_2|_p, \ldots, |\alpha_d|_p\}\big)\Big). \tag{6}$$

Then for $i = 1, \ldots, d$, we have $|x + \alpha_i|_p > 1$, and the $dm + 1$ elements of $\mathbb{Q}_p$:

$$1, G_p^{(2)}(x+\alpha_1), \ldots, G_p^{(m+1)}(x+\alpha_1), \ldots, G_p^{(2)}(x+\alpha_d), \ldots, G_p^{(m+1)}(x+\alpha_d)$$

are linearly independent over $\mathbb{Q}$.

Theorem 1.5 combined with (4) yields the following consequence.

Theorem 1.6. *Let d, m, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, p and x satisfying the hypotheses of Theorem 1.5. Then 1 together with the dm elements of $\mathbb{Q}_p$*

$$\omega(x+\alpha_i)^{1-s}\zeta_p(s, x+\alpha_i) \quad (1 \leq i \leq d \quad \text{and} \quad 2 \leq s \leq m+1)$$

are linearly independent over $\mathbb{Q}$, where ω denotes the Teichmüller character on $\mathbb{Q}_p^\times$.

Remark 1.7. In Section 11 we will prove more general statements -without the assumption that the denominators of $x, \alpha_2, \ldots, \alpha_d$ are powers of p- with an explicit condition instead of (6), see Theorems 11.1 and 11.2.

Remark 1.8. Note that according to [12, Proposition 11.2.9], given an integer $s \geq 2$ and $x \in \mathbb{Q}_p$ with $|x|_p \geq q_p$, we have

$$\zeta_p(s, x+1) - \zeta_p(s, x) = -\omega(x)^{s-1}x^{-s}.$$

In particular, if $x \in \mathbb{Q}$ is such that $\omega(x)^{s-1} \in \mathbb{Q}$ (which is always the case if $p-1$ divides $s-1$ for example), then $1, \zeta_p(s, x), \zeta_p(s, x+1)$ are linearly dependent over $\mathbb{Q}$. Condition (5) appearing in Theorems 1.5 and 1.6 is therefore necessary and quite natural.

We can deduce Theorem 1.2 (resp. Theorem 1.3) from the explicit version of Theorem 1.6 by choosing the parameters $d = 1$ and $x = p^{-r}$ (resp. $d = 2$, $x = p^{-a}$ and $\alpha_2 = p^{-b}$). In that case we will see that $\omega(p^{-r}) \in \mathbb{Q}$ (resp. $\omega(p^{-a}) = \omega(p^{-a} + p^{-b}) \in \mathbb{Q}$), see (15).

Our strategy. The proof of Theorem 1.5 is based on Padé approximants of second kind. This a similar approach to Beukers in [7, Theorem 9.2], although we will use different tools in a more general context. Our constructions heavily rely on formal integration transforms φ_f (see Section 4). This method was employed, though expressed differently, in [13, 14, 15, 25, 23]. While holonomic series were considered in the aforementioned studies, in this paper we examine their "difference analogs", in other words, formal Laurent series which satisfy a difference equation. For each integer $s \geq 2$, define the formal Laurent series $R_s(z)$ by

$$R_s(z) = \sum_{k=s-2}^{\infty} (k-s+3)_{s-2} B_{k-s+2} \cdot \frac{(-1)^{k+1}}{z^{k+1}},$$

where B_k denotes the k-th Bernoulli number and $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol. Then, for each $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, the s-th polygamma function $G_p^{(s)}$ evaluated at x is equal to $-R_s(x)$. The Laurent series $R_s(z)$ is the image of a G-function in the sense of Siegel [43] under a modified Mellin transform $\mathfrak{M}^{\mathrm{Inv}}$, whose definition is inspired by [4, Définition 1] and [2, Section 7]. This will allow us to show that $R_s(z)$ is a solution to a certain difference equation, see Proposition 5.4. We will then construct Padé approximants of second kind $(P_n(z), Q_{n,i,s}(z))_{(i,s)}$ of the series $(R_s(z+\alpha_i))_{(i,s)}$. This uses a difference analogue of the Rodrigues formula for Padé approximants, as outlined in [23, Section 2] by the first author.

To prove the main theorem using Siegel's method [43], it is necessary to demonstrate the non-vanishing of the determinant formed by those Padé approximants, which involves Bernoulli numbers. In previous works such as [13, 14, 15, 25, 23], this step is done by computing a "closed form" of the involved determinants, which can be quite a difficult and challenging problem in general, see [24] for example. Here, we develop new tools which allows us to prove the non-vanishing property rather "simply", without having to obtain such explicit formula. This approach is expected to apply in different settings. The rest of the proof is classical, although quite technical. Given a rational number x, we estimate the growth of the sequences $(|P_n(x)|)_n$,

$(|Q_{n,i,s}(x)|)_n$, $(|P_n(x)R_s(x+\alpha_i) - Q_{n,i,s}(x)|_p)_n$, as well as that of $(|D_n(x)|)_n$ and $(|D_n(x)|_p)_n$, where $D_n(x)$ denotes the common denominator of $P_n(x)$ and $Q_{n,i,s}(x)$. Suitable growth conditions ensure that the p-adic numbers $R_s(x+\alpha_i)$ together with 1 are linearly independent. For the estimates of $(|P_n(x)|)_n$, $(|Q_{n,i,s}(x)|)_n$, we explain in Section 10 how we can use Perron's second Theorem to improve the rough estimates that we get. To our knowledge, the majority of the Diophantine results based on Poincaré-Perron theorem involve recurrences of order 2. In our case, the order of the Poincaré-Perron recurrences involved is equal to $d(m+1)$, where the parameters d, m are as in Theorem 1.5. In particular, it can be strictly larger than 2.

Outline of our article. In Section 2 we introduce our notation and recall the definitions and some properties of Diamond's p-adic polygamma functions $G_p^{(s)}(z)$ and the p-adic Hurwitz zeta functions $\zeta_p(s,x)$. Several formal transforms will play an important role in our constructions, such as a modified inverse Mellin transform and formal integration transforms introduced in Section 3 and 4 respectively. In Section 5 we define some formal series $R_{\alpha,s}(z)$ which are related to the p-adic polygamma functions. Using the modified inverse Mellin transform, the formal integration transforms, and some basic properties of the difference operator established in Section 6, we construct some Padé approximants of the functions $R_{\alpha,s}(z)$ is Section 7. To prove our main theorem, we need to study these Padé approximants in more depth. First, as explained previously, we need to prove that they are linearly independent, which amounts to showing the non-vanishing of some determinant. This is a consequence of the main result of Section 8. Secondly, we need to establish several estimates, such as the growth of our Padé approximants and their denominators, as well as the p-adic growth of the Padé approximations. This is done in Section 9 and 10. Finally, Section 11 is devoted to the proof of more general and explicit versions of Theorems 1.5 and 1.6, whereas we prove refined and explicit versions of Theorems 1.2 and 1.3 in Section 12.

2 Notation

In this section, we introduce some notation and we give the definition of the log gamma function of Diamond $G_p(z)$, Diamond's p-adic polygamma functions and the p-adic Hurwitz zeta function $\zeta_p(s,x)$, as well as some basic properties they satisfy and that we will use later. In subsection 2.2, we recall some elements of Padé approximation theory.

2.1 The p-adic Hurwitz zeta function

The floor (resp. ceiling) function is denoted by $\lfloor \cdot \rfloor$ (resp. $\lceil \cdot \rceil$). For any $a \in \mathbb{Z}$, we denote by $\mathbb{Z}_{\leq a}$ the set of integers n with $n \leq a$. We define similarly $\mathbb{Z}_{<a}$, $\mathbb{Z}_{\geq a}$ and $\mathbb{Z}_{>a}$. The rising factorials are the polynomials $(x)_n = x(x+1)\cdots(x+n-1)$ $(n \in \mathbb{Z}_{\geq 0})$, with the convention that $(x)_0 = 1$. Given a (unitary) ring R and an integer $n \geq 1$, we denote by $R^\times$ the unit group of R.

In the following, we fix a prime number p and we set

$$q_p = \begin{cases} p & \text{if } p \geq 3 \\ 4 & \text{if } p = 2. \end{cases} \tag{7}$$

As usual, $\mathbb{Q}_p$ is the field of p-adic numbers, and $\mathbb{C}_p$ is the p-adic completion of an algebraic closure of $\mathbb{Q}_p$. We write $|\cdot|_p : \mathbb{C}_p \longrightarrow \mathbb{R}_{\geq 0}$ for the p-adic norm with the normalization $|p|_p = p^{-1}$. We denote by $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ the valuation which extends the usual p-adic valuation on $\mathbb{Z}$. With this notation, we have $|x|_p = p^{-v_p(x)}$ for each $x \in \mathbb{C}_p$. The ring of p-adic integers is the subset $\mathbb{Z}_p = \{x \in \mathbb{Q}_p\, ;\, |x|_p \leq 1\}$, and the group of units of $\mathbb{Z}_p$ is $\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p\, ;\, |x|_p = 1\}$.

Given $d \in \mathbb{Z}_{>0}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Q}^d$, define $\mu(\boldsymbol{\alpha})$ and the denominator $\mathrm{den}(\boldsymbol{\alpha})$ by

$$\mathrm{den}(\boldsymbol{\alpha}) = \min\{n \in \mathbb{Z}_{\geq 1}\,;\, n\alpha_i \in \mathbb{Z} \text{ for all } i = 1, \ldots, d\},$$
$$\mu(\boldsymbol{\alpha}) = \mathrm{den}(\boldsymbol{\alpha}) \prod_{\substack{q:\text{prime} \\ q \mid \mathrm{den}(\boldsymbol{\alpha})}} q^{\frac{1}{q-1}}. \tag{8}$$

Note that $\mathrm{den}(\boldsymbol{\alpha}) = \mathrm{lcm}\big(\mathrm{den}(\alpha_1), \ldots, \mathrm{den}(\alpha_d)\big)$, where lcm stands for the least common multiple. Thus $\mathrm{den}(\alpha_i) \leq \mathrm{den}(\boldsymbol{\alpha})$ and $\mu(\alpha_i) \leq \mu(\boldsymbol{\alpha})$ for $i = 1, \ldots, d$.

Bernoulli numbers and polynomials. We define the Bernoulli polynomials $B_n(x)$ by their exponential generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!}, \tag{9}$$

and the Bernoulli numbers B_n by $B_n = B_n(0)$. Recall that $B_n'(x) = nB_{n-1}(x)$ for each $n \geq 1$, and that $B_n(x)$ is a monoic polynomial of degree n. We also have the classic formulas $B_n(x+1) = B_n(x) + nx^{n-1}$, as well as

$$\sum_{k=0}^{n} \binom{n}{k} y^{n-k} B_k(x) = B_n(x+y) \quad \text{and} \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k(x) = nx^{n-1}.$$

For any positive integer k, Staudt-Clausen Theorem (see [11]) ensures that

$$B_{2k} + \sum_{p-1 \mid 2k} \frac{1}{p} \in \mathbb{Z} \tag{10}$$

where it is understood that p is a prime number. In particular, for any prime number p and any integer $n \geq 0$, we have $|B_n|_p \leq p$. It follows that for any $\alpha \in \mathbb{C}_p$, we have

$$|B_p(\alpha)|_p = \left| \sum_{k=0}^{n} \binom{n}{k} B_k \alpha^{n-k} \right|_p \leq p \cdot \max\{1, |\alpha|_p\}^n. \tag{11}$$

Volkenborn integral. A detailed study of the Volkenborn integral would be quite long, so we refer to [40] for the missing details. We say that a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$ is *Volkenborn integrable* if the sequence

$$p^{-n} \sum_{k=0}^{p^n - 1} f(k)$$

converges p-adically. In that case we call its limit the *Volkenborn integral* of f and we write

$$\int_{\mathbb{Z}_p} f(t)dt := \lim_{n \to \infty} p^{-n} \sum_{k=0}^{p^n - 1} f(k)$$

(*confer* [48]). For example, continuously differentiable functions and locally analytic functions are Volkenborn integrable, see [42, §55]. For all $x \in \mathbb{Q}_p$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$\int_{\mathbb{Z}_p} (x+t)^n dt = B_n(x)$$

(see [12, Section 11.1]).

Log gamma and polygamma functions of Diamond. We denote the (Iwasawa) p-adic logarithm function by $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$. The following properties characterize $\log_p$. We have $\log_p(xy) = \log_p(x)+\log_p(y)$ for each $x, y \in \mathbb{C}_p^\times$, $\log_p(p) = 0$ and

$$\log_p(1+x) = \sum_{n\geq 0} \frac{(-1)^{n+1}x^n}{n}$$

for each $x \in \mathbb{C}_p$ with $|x|_p < 1$. In [16], J. Diamond introduced a p-adic analog G_p of the classical $\log \Gamma$-function as follows.

Definition 2.1. The *log gamma function of Diamond* $G_p : \mathbb{C}_p \setminus \mathbb{Z}_p \to \mathbb{C}_p$ is the function given for each $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ by

$$G_p(x) = \int_{\mathbb{Z}_p} \Big((x+t)\log_p(x+t) - (x+t)\Big)dt.$$

For each integer $s \geq 1$, we also define $R_s : \mathbb{C}_p \setminus \mathbb{Z}_p \to \mathbb{C}_p$ by setting

$$R_s(x) = -G_p^{(s)}(x) \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p). \tag{12}$$

Recall that G_p satisfies the functional equation $G_p(x+1) = G_p(x) + \log_p(x)$ for each $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, and is locally analytic on $\mathbb{C}_p \setminus \mathbb{Z}_p$. Furthermore, $G_p(z)$ has the following expansion (*confer* [16, Theorem 6]):

$$G_p(z) = \left(z - \frac{1}{2}\right)\log_p(z) - z + \sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)} \cdot \frac{1}{z^k} \qquad (z \in \mathbb{C}_p, |z|_p > 1). \tag{13}$$

The successive derivatives of G_p are called the *Diamond's p-adic polygamma functions*. For our purpose, it is more convenient to work with R_s. By (13), the function R_2 has the following expansion:

$$R_2(z) = \sum_{k=0}^{\infty} B_k \cdot \frac{(-1)^{k+1}}{z^{k+1}} = \frac{B_0}{z} - \frac{B_1}{z^2} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \frac{B_6}{z^7} + \cdots \qquad (z \in \mathbb{C}_p, |z|_p > 1).$$

More generally, for each integer $s \geq 2$ and each $z \in \mathbb{C}_p$ with $|z|_p > 1$, we have

$$R_s(z) = R_2^{(s-2)}(z) = \sum_{k=s-2}^{\infty} (k-s+3)_{s-2} B_{k-s+2} \cdot \frac{(-1)^{k+1}}{z^{k+1}}. \tag{14}$$

Teichmüller character. Recall that q_p is defined in (7). We define the Teichmüller character $\omega : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ as follows. For any $x \in \mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p \,;\, |x|_p = 1\}$, we denote by $\omega(x)$ the unique $\varphi(q_p)$-th root of unity such that

$$\langle x \rangle := \frac{x}{\omega(x)} \in 1 + q_p\mathbb{Z}_p,$$

where φ is Euler's totient function. Then, given $y \in \mathbb{Q}_p^\times$, we put

$$\omega(y) = p^{v_p(y)}\omega\Big(p^{-v_p(y)}y\Big) \quad \text{and} \quad \langle y \rangle = \frac{y}{\omega(y)} = \langle p^{-v_p(y)}y \rangle \in 1 + q_p\mathbb{Z}_p.$$

Remark 2.2. There is a canonical isomorphism $\mathbb{Z}_p^\times \cong \mathcal{R}_p \times (1 + q_p\mathbb{Z}_p)$, which is precisely given by $x \mapsto (\omega(x), \langle x \rangle)$, were $\mathcal{R}_p$ denotes the subgroup of $\varphi(q_p)$-th unit roots of $\mathbb{Z}_p$. If $p \geq 3$, we have

$$\omega(x) = \lim_{n\to\infty} x^{p^n},$$

and $\omega(x)$ is the unique $(p-1)$-th root of unity that is congruent to x modulo p (see [42, Theorem 33.4]). We deduce easily from the above that

$$\omega(p^{-a}+y) = p^{-a}\omega(1+yp^{a}) = p^{-a} \in \mathbb{Q} \tag{15}$$

for each integer $a \geq 0$ and each $y \in \mathbb{Q}_p$ with $|y|_p < p^a$. In the special case $p = 2$, either x or $-x$ is congruent to 1 modulo $4 = q_p$, and we set $\omega(x) = \pm 1$, so that x is congruent to $\omega(x)$ modulo 4. Note that this last definition differs from [42, Definition 33.3] (which would give $\omega(x) = 1$ for each $x \in \mathbb{Z}_2^\times$).

The p-adic Hurwitz zeta function. Recall that q_p is defined in (7). We follow [12, Definition 11.2.5] for the definition of the p-adic Hurwitz zeta function $\zeta_p(s,x)$.

Definition 2.3. For $x \in \mathbb{Q}_p$ and $s \in \mathbb{C}_p \setminus \{1\}$ with $|x|_p \geq q_p$ and $|s|_p < q_p p^{-1/(p-1)}$, we define $\zeta_p(s,x)$ by the equivalent formulas

$$\zeta_p(s,x) = \frac{1}{s-1}\int_{\mathbb{Z}_p} \langle x+t\rangle^{1-s} dt = \frac{\langle x\rangle^{1-s}}{s-1}\sum_{k\geq 0}\binom{1-s}{k} B_k x^{-k}.$$

For a fixed $x \in \mathbb{Q}_p$ with $|x|_p \geq q_p$, the function $s \mapsto \zeta_p(s,x)$ is the unique p-adic meromorphic function on $|s|_p < q_p p^{-1/(p-1)}$ satisfying

$$\zeta_p(1-n,x) = -\omega(x)^{-n}\frac{B_n(x)}{n}$$

for each integer $n \geq 1$. In addition, this function is analytic, except for a simple pole at $s = 1$ with residue 1 (see [12, Proposition 11.2.8]). Note that for $p \geq 3$ the condition "$x \in \mathbb{Q}_p$ with $|x|_p \geq q_p$" simply means that $x \in \mathbb{Q}_p \setminus \mathbb{Z}_p$.

The following identity (see [12, Proposition 11.5.6.]) (which is equivalent to (4)) combined with Theorem 1.5 implies Theorem 1.6.

$$\omega(x)^{1-s}\zeta_p(s,x) = \frac{(-1)^s}{(s-1)!}G_p^{(s)}(x) = \frac{(-1)^{s+1}}{(s-1)!}R_s(x) \qquad (x \in \mathbb{Q}_p, |x|_p \geq q_p). \tag{16}$$

2.2 Padé approximation theory

Fix a field K of characteristic 0. For any subset S of a K-vector space V, we denote by $\langle S\rangle_K \subseteq V$ the K-vector space generated by the elements of S. Given an integer $n \geq 0$, we denote by $K[z]$ the ring of polynomials in z with coefficients in K, and by $K[z]_{\leq n} \subseteq K[z]$ the subgroup of polynomials of degree at most n.

Let us recall the definition of Padé-type approximants of Laurent series and their basic properties. We denote by $K[[1/z]]$ the ring of formal power series ring of variable $1/z$ with coefficients in K, and by $K((1/z))$ its field of fractions. We say that an element of $K((1/z))$, which can be written as

$$\sum_{k=-n}^{\infty}\frac{a_k}{z^k},$$

with $n \in \mathbb{Z}$ and $a_k \in K$, is a *formal Laurent series* in $1/z$. We define the *order* function at $z = \infty$ by

$$\mathrm{ord}_\infty : K((1/z)) \longrightarrow \mathbb{Z}\cup\{\infty\};\ \sum_k \frac{a_k}{z^k} \mapsto \min\{k \in \mathbb{Z}\cup\{\infty\}\,;\, a_k \neq 0\}.$$

with the convention $\min\emptyset = \infty$. In particular, for any $f \in K((1/z))$, we have $\mathrm{ord}_\infty f = \infty$ if and only if $f = 0$.

Given two K-vector spaces V and W, we denote by $\mathrm{Hom}_K(V,W)$ the K-vector space of K-linear homomorphisms $V \to W$. When $V = W$, we write $\mathrm{End}_{K\text{-lin}}(V) = \mathrm{Hom}_K(V,W)$. Similarly, for any K-algebra $\mathcal{A}$, we define $\mathrm{End}_{K\text{-alg}}(\mathcal{A})$ as the K-vector space of K-algebra endomorphisms of $\mathcal{A}$.

We recall, without proof, the following basic result from Padé approximation theory.

LEMMA 2.4. *Let m be a positive integer, $f_1,\dots,f_m \in (1/z)\cdot K[[1/z]]$ and $\boldsymbol{n} = (n_1,\dots,n_m) \in \mathbb{N}^m$. Put $N = \sum_{j=1}^m n_j$. For any non-negative integer $M \geq N$, there exists a non-zero vector of polynomials $(P,Q_1,\dots,Q_m) \in K[z]^{m+1}$ satisfying the following conditions:*

(i) $\deg P \leq M$,

(ii) $\mathrm{ord}_\infty\,(P(z)f_j(z) - Q_j(z)) \geq n_j + 1$ *for* $j = 1,\dots,m$.

DEFINITION 2.5. With the notation of Lemma 2.4, fix a non-zero vector of polynomials $(P,Q_1,\dots,Q_m) \in K[z]^{m+1}$ satisfying the properties (i) and (ii).

- We say that $(P,Q_1,\dots,Q_m) \in K[z]^{m+1}$ is a *weight $\boldsymbol{n}$ and degree M Padé-type approximant of* $(f_1,\dots,f_m)$.
- We call the remainders, namely the formal Laurent series $(P(z)f_j(z) - Q_j(z))_{1\leq j\leq m}$, *weight $\boldsymbol{n}$ degree M Padé-type approximations of* $(f_1,\dots,f_m)$.

3 Modified formal transforms

Fix a field K of characteristic 0. We introduce a modified inverse Mellin transform $\mathfrak{M}_K^{\mathrm{Inv}}$ for formal power series, and we study some of its properties. This transform will play a key-role in studying the properties of the explicit Padé approximants constructed in Section 7. In the next section, we will compute the inverse Mellin transform of some formal series connected to polygamma functions.

Formal series (examples). Fix $\alpha \in K$. We define the following formal series of $K[[z]]$ as usual

$$\exp(z) = \sum_{n\geq 0} \frac{z^n}{n!}, \quad \log(1+z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} z^n \quad \text{and} \quad (1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k,$$

where $\binom{\alpha}{k} = \alpha(\alpha-1)\cdots(\alpha-k+1)/k!$ (with the convention that $\binom{\alpha}{0} = 1$). We also see $1/(z+\alpha)$ as an element of $(1/z)\cdot K[[1/z]]$ by writing

$$\frac{1}{z+\alpha} = \frac{1}{z}\sum_{n\geq 0}\left(-\frac{\alpha}{z}\right)^n.$$

Difference and differential operators of $K[[1/z]]$. Given $\alpha \in K$ we denote by τ_α the α-shift operator, and by $\Delta_\alpha = \tau_\alpha - 1$ the α-difference operator of $K((1/z))$. Note that they stabilize $K[[1/z]]$ and $(1/z)\cdot K[[1/z]]$. They are defined for each $f(z) \in K((1/z))$ by

$$\tau_\alpha\big(f(z)\big) = f(z+\alpha) \quad \text{and} \quad \Delta_\alpha\big(f(z)\big) = f(z+\alpha) - f(z).$$

Fix $f(z) \in K((1/z))$ and a sequence $(a_n)_{n\geq 0}$ of elements in K. Then, the series

$$\sum_{n\geq 0} a_n \Delta_\alpha^n\big(f(z)\big) \quad \text{and} \quad \sum_{n\geq 0} a_n \frac{d^n}{dz^n} f(z)$$

converge in $K((1/z))$, since for each integer $n \geq 0$, the formal series $\Delta_\alpha^n(f(z))$ and $\frac{d^n}{dz^n} f(z)$ belong to $(1/z)^{n+d} K[[1/z]]$, where $d = \mathrm{ord}_\infty(f) \in \mathbb{Z}$. The sets $K[[\Delta_\alpha]]$ and $K[[d/dz]]$ are therefore subsets of the set of $\mathrm{End}_{K\text{-lin}}\big(K((1/z))\big)$. The above argument shows that they are also subsets of $\mathrm{End}_{K\text{-lin}}\big((1/z)\cdot K[[1/z]]\big)$ and $\mathrm{End}_{K\text{-lin}}(K[[1/z]])$. The following result will be useful.

Lemma 3.1. *For any $\alpha \in K$, we have*

$$\exp\left(\alpha \frac{d}{dz}\right) = \tau_\alpha \quad \textit{and} \quad \exp\left(\frac{d}{dz}\right) - 1 = \Delta_1. \tag{17}$$

So $K[[d/dz]] = K[[\Delta_1]]$,

$$\log(1+\Delta_1) = \frac{d}{dz} \quad \textit{and} \quad (1+\Delta_1)^\alpha = \tau_\alpha. \tag{18}$$

Proof. Since $\Delta_1 = \tau_1 - 1$, we only have to prove the first equality in (17). Write

$$D := \exp\left(\alpha\frac{d}{dz}\right) = \sum_{k\geq 0} \frac{\alpha^k}{k!}\frac{d^k}{dz^k}.$$

Let $n \geq 0$ be an integer. A quick computation yields

$$D\left(\frac{1}{z^n}\right) = \sum_{k\geq 0} \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}\frac{\alpha^k}{z^{n+k}} = \frac{1}{z^n}\frac{1}{(1+\alpha/z)^n} = \frac{1}{(z+\alpha)^n} = \tau_\alpha\left(\frac{1}{z^n}\right),$$

hence (17). □

Formal Laplace transform and Stirling numbers. Following [8, p. 154], we define the (formal) modified Laplace transform

$$\mathfrak{L}_K : K[[z]] \longrightarrow K[[z]]; \quad \sum_{k=0}^\infty a_k \frac{z^k}{k!} \mapsto \sum_{k=0}^\infty a_k z^k.$$

Then $\mathfrak{L}_K$ is a homeomorphism of $K[[z]]$ (with respect to the (z)-adic topology).
Given a pair of non-negative integers (k,n), we define $\mathcal{S}(n,k)$ as the number of ways of partitioning a set of n elements into k non-empty sets. The numbers $\mathcal{S}(n,k)$ (sometimes denoted by $\left\{ n \atop k \right\}$, see [26]) are called *Stirling number of the second kind* [46]. They satisfy the recurrence relation

$$\mathcal{S}(n,k) = \mathcal{S}(n-1,k-1) + k\mathcal{S}(n-1,k) \qquad (k,n\geq 1) \tag{19}$$

with initial conditions $\mathcal{S}(0,0) = 1$ and $\mathcal{S}(n,0) = \mathcal{S}(0,n) = 0$ for each $n \geq 1$. They also have the following generating functions

$$\frac{(e^z-1)^n}{n!} = \sum_{k=n}^\infty \mathcal{S}(k,n)\frac{z^k}{k!} \quad \text{and} \quad \frac{z^n}{(1-z)\cdots(1-nz)} = \sum_{k=n}^\infty \mathcal{S}(k,n) z^k \tag{20}$$

for each integer $n \geq 0$ (see for example [32, Chapter V, §26-27]). The above identities can be obtained by using (19). We deduce from (20) the following result.

Lemma 3.2. *Let n be a non-negative integer. Then,*

$$\mathfrak{L}_K\left(\frac{(e^z-1)^n}{n!}\right) = \frac{z^n}{(1-z)\cdots(1-nz)}.$$

Formal modified Mellin transform. The definition of our modified Mellin transforms are inspired by [4, Définition 1] and [2, Section 7], see Remark 3.7 below. Recall that by Lemma 3.1, we have $K[[d/dz]] = K[[\Delta_1]]$.

Definition 3.3. The correspondence $z \mapsto \Delta_1 = \exp(d/dz) - 1$ defines an isomorphism of K-algebra

$$\widehat{\mathfrak{M}}_K^{\mathrm{Inv}} : K[[z]] \longrightarrow K[[d/dz]].$$

When there is no ambiguity, we simply write $\widehat{\mathfrak{M}}^{\mathrm{Inv}} = \widehat{\mathfrak{M}}_K^{\mathrm{Inv}}$.

Example 3.4. We deduce from Lemma 3.1 that for any $\alpha \in K$, we have

$$\widehat{\mathfrak{M}}^{\mathrm{Inv}}\big((1+z)^{\alpha}\big) = \tau_{\alpha} \quad \text{and} \quad \widehat{\mathfrak{M}}^{\mathrm{Inv}}\big(\log(1+z)\big) = \frac{d}{dz}.$$

Definition 3.5. We call *modified inverse Mellin transform* of power series the map

$$\mathfrak{M}_K^{\mathrm{Inv}} : K[[z]] \longrightarrow (1/z)\cdot K\left[[1/z]\right]; \quad g(z) \mapsto \sum_{k=0}^{\infty} b_k \left(-\frac{1}{z}\right)^{k+1},$$

where $(b_k)_{k\geq 0}$ is defined by $g(e^z - 1) = \sum_{k=0}^{\infty} b_k z^k/k!$. When there is no ambiguity, we simply write $\mathfrak{M}^{\mathrm{Inv}} = \mathfrak{M}_K^{\mathrm{Inv}}$.

The transform $\mathfrak{M}^{\mathrm{Inv}}$ satisfies the following property.

Proposition 3.6. *Let* $g(z) = \sum_{k=0}^{\infty} a_k z^k \in K[[z]]$. *Then,*

$$\mathfrak{M}^{\mathrm{Inv}}(g)(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} a_k k!}{z(z+1)\cdots(z+k)}.$$

Proof. Define the morphism

$$\mathfrak{T}_K : K[[z]] \longrightarrow (1/z)\cdot K\left[[1/z]\right]; \quad f(z) \longmapsto -\frac{1}{z} f\Big(-\frac{1}{z}\Big),$$

so that $\mathfrak{M}^{\mathrm{Inv}}(g)(z) = \mathfrak{T}_K \circ \mathfrak{L}_K\big(g(e^z-1)\big)$. By Lemma 3.2, we find

$$\mathfrak{L}_K\big(g(e^z-1)\big) = \sum_{k=0}^{\infty} a_k \mathfrak{L}_K\big(g(e^z-1)^k\big) = \sum_{k=0}^{\infty} \frac{a_k k! z^k}{(1-z)\cdots(1-kz)}.$$

Then conclusion follows easily. □

Remark 3.7. The ring of inverse factorial series (with complex coefficients) is

$$\mathbb{C}[!z!] := \left\{ \sum_{n\geq 0} \frac{a_n}{z(z+1)\cdots(z+n)}\ ;\ a_n \in \mathbb{C} \right\},$$

(it corresponds to the set $\mathbb{C}[!z!]^{(0)}$ with the notation of [2, Section 7.2]). Note that $\mathbb{C}[!z!] = (1/z)\cdot\mathbb{C}[[1/z]]$. In [2, Section 7.2] and [4, Section 1.2], the authors consider the formal Mellin transform

$$\mathcal{M} : \mathbb{C}[!z!] \longrightarrow \mathbb{C}[[1-z]]; \quad \sum_{n\geq 0} \frac{a_n}{z(z+1)\cdots(z+n)} \longmapsto \sum_{n\geq 0} a_n \frac{(1-z)^n}{n!}.$$

Proposition 3.6 implies that $\mathfrak{M}_{\mathbb{C}}^{\mathrm{Inv}} = -\mathcal{M}^{-1} \circ \mathfrak{T}_{-1}$, where $\mathfrak{T}_{-1} : \mathbb{C}[[z]] \to \mathbb{C}[[1-z]]$ is the isomorphism given by $z \mapsto z-1$.

We end this section with an analog of [4, Proposition 3], which establishes a relation between the formal transforms $\widehat{\mathfrak{M}}^{\mathrm{Inv}}$ and $\mathfrak{M}^{\mathrm{Inv}}$.

Proposition 3.8. *For any $f(z), g(z) \in K[[z]]$, we have*

$$\mathfrak{M}^{\mathrm{Inv}}\big(f(z)g(z)\big) = \widehat{\mathfrak{M}}^{\mathrm{Inv}}(f(z))\big(\mathfrak{M}^{\mathrm{Inv}}(g(z))\big). \tag{21}$$

This yields the following commutative diagram:

$$\begin{array}{ccc} K[[z]] \times K[[z]] & \xrightarrow{\widehat{\mathfrak{M}}^{\mathrm{Inv}} \times \mathfrak{M}^{\mathrm{Inv}}} & K\,[[d/dz]] \times (1/z) \cdot K\,[[1/z]] \\ \big\downarrow & & \big\downarrow \\ K[[z]] & \xrightarrow{\mathfrak{M}^{\mathrm{Inv}}} & (1/z) \cdot K\,[[1/z]]\,. \end{array}$$

Proof. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k \in K[[z]]$. Since $\Delta_1(1/(z)_{k+1}) = -(k+1)/(z)_{k+2}$ for each integer $k \geq 0$, we have

$$\begin{aligned} \widehat{\mathfrak{M}}^{\mathrm{Inv}}(z)(\mathfrak{M}^{\mathrm{Inv}}(g)(z)) = \Delta_1 \left(\sum_{k=0}^{\infty} \frac{(-1)^{k+1} a_k k!}{z(z+1)\cdots(z+k)} \right) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+2} a_k (k+1)!}{z(z+1)\cdots(z+k+1)} \\ &= \mathfrak{M}^{\mathrm{Inv}}(zg)(z), \end{aligned}$$

the last equality coming from Proposition 3.6. By induction, we obtain $\widehat{\mathfrak{M}}^{\mathrm{Inv}}(z^n)(\mathfrak{M}^{\mathrm{Inv}}(g)(z)) = \mathfrak{M}^{\mathrm{Inv}}(z^n g)(z)$ for each integer $n \geq 1$. Hence (21) (by linearity). ☐

4 Formal f-integration transform

We keep the notation of Sections 3. In this section, we introduce and study the properties of the transform φ_f, which will play a crucial role in the construction of our Padé approximants (see [23, Section 2], [25, Section 3] for other applications related to those maps).

4.1 Notation and definitions

Definition 4.1. We associate to any Laurent series $f(z) = \sum_{k=0}^{\infty} f_k / z^{k+1} \in (1/z) \cdot K[[1/z]]$, a K-linear morphism

$$\varphi_f : K[t] \longrightarrow K; \quad t^k \mapsto f_k \quad (k \geq 0). \tag{22}$$

We call the map φ_f the *formal f-integration transform*.

Remark 4.2. In the case $f(z) = -\log(1 - 1/z)$, the map φ_f is simply the operator $P(t) \mapsto \int_0^1 P(u)du$, which is the reason why we call it f-integration transform.

Note that the K-linear map

$$\Phi : (1/z) \cdot K[[1/z]] \longrightarrow \mathrm{Hom}_K(K[t], K)$$

defined by $f \mapsto \varphi_f$ is an isomorphism. Given $f \in (1/z) \cdot K[[1/z]]$, the map φ_f extends naturally in a $K[z]$-linear map $\varphi_f : K[z,t] \to K[z]$, and then to a $K((1/z))$-linear map $\varphi_f : L \to z^n K[[1/z]]$, where L denotes ring which consists in all the elements of the form

$$F(z,t) = \sum_{k=-n}^{\infty} \frac{P_k(t)}{z^k},$$

with $n \in \mathbb{Z}$ and $P_k(t) \in K[t]$. Explicitly, for any element $F(z,t)$ as above, we have

$$\varphi_f(F(z,t)) = \sum_{k \geq -n} \frac{\varphi_f(P_k(t))}{z^k} \in z^n K[[1/z]].$$

With this notation, and seeing $1/(z-t) = \sum_{k\geq 1} t^k/z^{k+1}$ as an element of L, the formal Laurent series $f(z)$ satisfies the following crucial identities (*confer* [33, (6.2) p.60 and (5.7) p.52]):

$$f(z) = \varphi_f\left(\frac{1}{z-t}\right), \quad P(z)f(z) - \varphi_f\left(\frac{P(z)-P(t)}{z-t}\right) \in (1/z)\cdot K[[1/z]],$$

for any $P(z) \in K[z]$. Let us recall a condition, based on the morphism φ_f, for given polynomials to be Padé approximants.

Lemma 4.3 (*confer* [23, Lemma 2.3]). *Let m, M be positive integers, $f_1(z),\dots,f_m(z) \in (1/z)\cdot K[[1/z]]$ and $\boldsymbol{n} = (n_1,\dots,n_m) \in \mathbb{N}^m$ with $\sum_{j=1}^m n_j \leq M$. Let $P(z) \in K[z]$ be a non-zero polynomial with $\deg P \leq M$, and put $Q_j(z) = \varphi_{f_j}((P(z)-P(t))/(z-t)) \in K[z]$ for $1 \leq j \leq m$. The following assertions are equivalent.*

(i) The vector of polynomials $(P, Q_1, \dots, Q_m)$ is a weight $\boldsymbol{n}$ Padé-type approximants of $(f_1,\dots,f_m)$.

(ii) We have $t^k P(t) \in \ker \varphi_{f_j}$ for any pair of integers (j,k) with $1 \leq j \leq m$ and $0 \leq k \leq n_j - 1$.

Notation. In the next section, we will use the following operators. We use bold symbols to indicate when a map is defined on the ring $K[t]$ (in order to distinguish them from their analogues defined on $K[z]$ in Section 3). Fix an element $\alpha \in K$.

- $[P] \in \mathrm{End}_{K\text{-lin}}(K[t])$ denotes the multiplication by the polynomial $P \in K[t]$. If there is no ambiguity, we will sometimes omit the brackets.
- We denote by $\mathrm{Eval}_{t=\alpha} \in \mathrm{Hom}_K(K[t],K)$ the α-evaluation linear form, and by $\boldsymbol{\tau}_\alpha \in \mathrm{End}_{K\text{-alg}}(K[t])$ (resp. $\boldsymbol{\Delta}_\alpha = \boldsymbol{\tau}_\alpha - [1] \in \mathrm{End}_{K\text{-lin}}(K[t])$) the α-shift (resp. α-difference) operator. They are given by

$$\mathrm{Eval}_{t=\alpha}(P(t)) = P(\alpha), \quad \boldsymbol{\tau}_\alpha(P(t)) = P(t+\alpha) \quad \text{and} \quad \boldsymbol{\Delta}_\alpha(P(t)) = P(t+\alpha) - P(t),$$

  for each $P(t) \in K[t]$.
- For any $f(z) \in K((1/z))$, we denote by $\pi(f(z))$ the unique element of $(1/z)\cdot K[[1/z]]$ such that $f(z) - \pi(f(z)) \in K[z]$. This defines a (surjective) projection

$$\pi : K((1/z)) \longrightarrow (1/z)\cdot K[[1/z]]. \tag{23}$$

- The map $\iota : K[z, \tau_\alpha] \longrightarrow K[t, \boldsymbol{\tau}_{-\alpha}]$ is defined for each $D = \sum_i [a_i(z)] \circ \tau_\alpha \in K[z,\tau_\alpha]$ by

$$D^* := \iota\Big(\sum_i [a_i(z)] \circ \tau_\alpha\Big) = \sum_i \boldsymbol{\tau}_{-\alpha} \circ [a_i(t)],$$

  (this is a similar operator as the one in [1, Exercise III(3)]).

For any $D \in K[z,\tau_\alpha]$, we view D and D^* as elements of $\mathrm{End}_{K\text{-lin}}(K[t][[1/z]])$ by setting for each $f(z,t) = \sum_{n\geq 0} P_k(t)/z^n \in K[t][[1/z]]$

$$D(f(z,t)) = \sum_{n\geq 0} P_k(t) D\left(\frac{1}{z^n}\right) \quad \text{and} \quad D^*(f(z,t)) = \sum_{n\geq 0} \frac{D^*(P_k(t))}{z^n}.$$

4.2 Properties of the operators D and D^*

In the following lemma, we consider $1/(z-t) = \sum_{k\geq 0} t^k/z^{k+1}$ as an element of $K[t][[1/z]]$.

LEMMA 4.4. *Let $\alpha \in K$ and $D \in K[z, \tau_\alpha]$. Then there exists a polynomial $P(t,z) \in K[t,z]$ such that*

$$D\left(\frac{1}{z-t}\right) = P(t,z) + D^*\left(\frac{1}{z-t}\right). \tag{24}$$

PROOF. By linearity, it suffices to prove the statement of $D = [z^m] \circ \tau_\alpha^n$, where m, n are non-negative integers. We have

$$\begin{aligned}\tau_\alpha^n\left(\frac{1}{z-t}\right) &= \tau_\alpha^n\left(\sum_{k=0}^{\infty}\frac{t^k}{z^{k+1}}\right) = \sum_{k=0}^{\infty}\frac{t^k}{(z+n\alpha)^{k+1}} \\ &= \sum_{k=0}^{\infty}\frac{t^k}{z^{k+1}}\left(\sum_{\ell=0}^{\infty}(-1)^\ell\binom{k+\ell}{k}\left(\frac{\alpha n}{z}\right)^\ell\right) = \sum_{k=0}^{\infty}\frac{(t-n\alpha)^k}{z^{k+1}},\end{aligned}$$

hence

$$D\left(\frac{1}{z-t}\right) = P(t,z) + \sum_{k=0}^{\infty}\frac{(t-n\alpha)^{k+m}}{z^{k+1}}, \quad \text{where } P(t,z) = \sum_{k=0}^{m-1}(t-n\alpha)^k z^{m-k-1}. \tag{25}$$

(with the convention that $P(t,z) = 0$ if $m = 0$). On the other hand

$$D^*\left(\frac{1}{z-t}\right) = \boldsymbol{\tau}_{-\alpha}^n \circ [t^m]\left(\frac{1}{z-t}\right) = \sum_{k=0}^{\infty}\frac{(t-n\alpha)^{k+m}}{z^{k+1}}.$$

Combining the above with (25), we obtain (24). This completes the proof of the lemma. ☐

4.3 Properties of the transform φ_f

Lemma 4.4 allows us to show the following key proposition.

PROPOSITION 4.5. *Let $f(z) \in (1/z)\cdot K[[1/z]]$ and $D \in K[z, \tau_\alpha]$. Then, viewing $\varphi_{\pi(D(f))}$ and $\varphi_f \circ D^*$ as elements of $\mathrm{Hom}_K(K[t], K)$, we have*

$$\varphi_{\pi(D(f))} = \varphi_f \circ D^*.$$

PROOF. Let $P(t,z) \in K[t,z]$ be such that (24) of Lemma 4.4 holds. Then, writing $P(z) = \varphi_f(P(t,z))$ and since φ_f acts only on the parameter t, we have

$$\begin{aligned}D(f(z)) = D\circ\varphi_f\left(\frac{1}{z-t}\right) = \varphi_f\left(D\left(\frac{1}{z-t}\right)\right) &= P(z) + \varphi_f\left(D^*\left(\frac{1}{z-t}\right)\right) \\ &= P(z) + \sum_{k=0}^{\infty}\frac{\varphi_f(D^*(t^k))}{z^{k+1}}.\end{aligned}$$

This shows that

$$\pi(D(f)) = \sum_{k=0}^{\infty}\varphi_f(D^*(t^k))/z^{k+1} \quad \text{and} \quad \varphi_{\pi(D(f))}(t^k) = \varphi_f\circ D^*(t^k) \quad \text{for all } k \geq 0.$$

This completes the proof of Proposition 4.5. ☐

We deduce several interesting consequences from Proposition 4.5.

Corollary 4.6. *Let $f(z) \in (1/z)\cdot K[[1/z]]$ and $D \in K[z,\tau_\alpha]\setminus\{0\}$. The following assertions are equivalent.*

(i) $D(f(z)) \in K[z]$.

(ii) $D^*(K[t]) \subseteq \ker \varphi_f$.

Proof. Conditions (i) and (ii) are equivalent to $\pi(D(f)) = 0$ and $\varphi_f \circ D^* = 0$, respectively. Those last two assertions are clearly equivalent by Proposition 4.5. □

Corollary 4.7. *Let $f(z) \in (1/z)\cdot K[[1/z]]$ and $\alpha \in K$, and set $g(z) = \tau_\alpha(f(z)) \in (1/z)\cdot K[[1/z]]$. Then*

$$\varphi_g = \varphi_f \circ \boldsymbol{\tau}_{-\alpha}.$$

Proof. As discussed at the beginning of Section 3, we have $g(z) \in (1/z)\cdot K[[1/z]]$, so that $\pi(g(z)) = g(z)$. Set $D = \tau_\alpha$. By definition $D^* = \boldsymbol{\tau}_{-\alpha}$, and we conclude by Proposition 4.5. □

We also deduce from Proposition 4.5 the following results.

Lemma 4.8. *Let $f \in (1/z)\cdot K[[1/z]]$ and write*

$$f(z) = \sum_{k\geq 0} \frac{a_k k!}{z(z+1)\cdots(z+k)}.$$

Then, for each $j \geq 0$, we have

$$\varphi_f\left(\frac{(t)_j}{j!}\right) = a_j.$$

Proof. Recall that $\pi : K[z][[1/z]] \to (1/z)\cdot K[[1/z]]$ is the morphism defined in (23). Fix an integer $j \geq 0$ and set $D = (z)_j/j!$. Then $D^* = (t)_j/j!$, and Proposition 4.5 gives

$$\varphi_f\left(\frac{(t)_j}{j!}\right) = \varphi_f(D^*(1)) = \varphi_{\pi(D(f))}(1). \tag{26}$$

On the other hand, since $(z)_j/(z)_{k+1} \in K[z]$ for $k = 0,\ldots,j-1$, we have

$$\pi(D(f)) = \frac{1}{j!}\sum_{k=j}^{\infty} \frac{a_k k!}{(z+j)(z+j+1)\cdots(z+k)} \in \frac{a_j}{z} + \frac{1}{z^2}K[[1/z]],$$

hence $\varphi_{\pi(D(f))}(1) = a_j$. Combined with (26), this proves the lemma. □

Combined with Proposition 3.6, the above lemma has the following consequence.

Corollary 4.9. *Let $g(z) = \sum_{k=0}^{\infty} a_k z^k \in K[[z]]$, and set $f(z) = \mathfrak{M}^{\mathrm{Inv}}(g)(z) \in (1/z)\cdot K[[1/z]]$. For each $j \geq 0$, we have*

$$\varphi_f\left(\frac{(t)_j}{j!}\right) = (-1)^{j+1}a_j.$$

5 Mellin transform and p-adic polygamma functions

We keep the notation of Section 3. Denote by $\mathfrak{M}^{\mathrm{Inv}} = \mathfrak{M}_K^{\mathrm{Inv}}$ the inverse modified Mellin transform (see Definition 3.5). In this section, we introduce and study a family of formal series $R_{\alpha,s}(z) \in (1/z)\cdot K[[1/z]]$ connected to the p-adic polygamma functions. We also prove that the series $R_{\alpha,s}$ satisfies a rather simple difference equation.

Definition 5.1. Given an integer $s \geq 2$ and $\alpha \in K$, set

$$R_{\alpha,s}(z) = \mathfrak{M}^{\mathrm{Inv}}\left(\frac{(1+z)^{\alpha}\log^{s-1}(1+z)}{z}\right),$$
$$R_{\alpha,1}(z) = \mathfrak{M}^{\mathrm{Inv}}\left(\frac{(1+z)^{\alpha}-1}{z}\right).$$

For $\alpha = 0$, we simply write

$$R_s(z) = R_{0,s}(z) = \mathfrak{M}^{\mathrm{Inv}}\left(\frac{\log^{s-1}(1+z)}{z}\right).$$

In the following lemma we show that $R_s(z) \in (1/z)\cdot K[[1/z]]$ is the formal series corresponding to the series (14) of the p-adic polygamma functions, which is why we are using the same notation.

Lemma 5.2. *With the notation of Definition 5.1, we have*

$$R_s(z) = \sum_{k=s-2}^{\infty}(k-s+3)_{s-2}B_{k-s+2}\cdot\frac{(-1)^{k+1}}{z^{k+1}},$$
$$R_{\alpha,s}(z) = \tau_{\alpha}(R_s(z)) = R_s(z+\alpha),$$
$$R_{\alpha,1}(z) = \sum_{k=0}^{\infty}\frac{B_{k+1}(\alpha)-B_{k+1}}{k+1}\cdot\frac{(-1)^{k+1}}{z^{k+1}}.$$

Furthermore

(i) The series $R_{\alpha,s}(x)$ converges p-adically for each $x,\alpha \in \mathbb{C}_p$ with $|x+\alpha|_p > 1$.

(ii) The series $R_{\alpha,1}(x)$ converges p-adically for each $x,\alpha \in \mathbb{C}_p$ with $|x|_p > |\alpha|_p$.

Proof. Write $g(z) = \log^{s-1}(1+z)/z$ so that $R_s(z) = \mathfrak{M}^{\mathrm{Inv}}(g)(z)$. Using the generating function of Bernoulli numbers (9), we find

$$g(e^z-1) = \frac{z^{s-1}}{e^z-1} = \sum_{k=s-2}^{\infty}(k-s+3)_{s-2}B_{k-s+2}\cdot\frac{z^k}{k!}.$$

The expansion of $R_s(z)$ follows from the definition of $\mathfrak{M}^{\mathrm{Inv}}$. The second identity is then a direct consequence of Lemma 5.3 below. For the third identity, write $h(z) = \big((1+z)^{\alpha}-1\big)/z$. Again, using (9), we find

$$h(e^z-1) = \frac{e^{\alpha z}-1}{e^z-1} = \frac{1}{z}\sum_{k=0}^{\infty}\big(B_k(\alpha)-B_k\big)\cdot\frac{z^k}{k!} = \sum_{k=0}^{\infty}\big(B_{k+1}(\alpha)-B_{k+1}\big)\cdot\frac{z^k}{(k+1)!}.$$

The expected formula follows from the definition of $\mathfrak{M}^{\mathrm{Inv}}$, since $R_{\alpha,1}(z) = \mathfrak{M}^{\mathrm{Inv}}(h)(z)$. The convergence in $\mathbb{C}_p$ of the series $R_{\alpha,s}(x)$ and $R_{\alpha,1}(x)$ is a consequence of (11). □

We deduce from Proposition 3.8 the following crucial identities.

Lemma 5.3. *Let $g(z) = \sum_{n\geq 0} a_n z^n \in K[[z]]$ and $\alpha \in K$. We have*

$$\mathfrak{M}^{\mathrm{Inv}}\big(\log(1+z)g\big)(z) = \frac{d}{dz}\big(\mathfrak{M}^{\mathrm{Inv}}(g)(z)\big),$$
$$\mathfrak{M}^{\mathrm{Inv}}\big((1+z)^{\alpha}g\big)(z) = \tau_{\alpha}(\mathfrak{M}^{\mathrm{Inv}}(g)(z)),$$
$$\mathfrak{M}^{\mathrm{Inv}}\big(((1+z)^{\alpha}-1)g\big)(z) = \Delta_{\alpha}(\mathfrak{M}^{\mathrm{Inv}}(g)(z)).$$

Proof. This follows from Proposition 3.8 combined respectively with

$$\widehat{\mathfrak{M}}^{\mathrm{Inv}}(\log(1+z)) = d/dz \quad \text{and} \quad \widehat{\mathfrak{M}}^{\mathrm{Inv}}((1+z)^{\alpha}) = \tau_{\alpha}$$

coming from Lemma 3.1. □

Proposition 5.4 (Difference equation of the Laurent series $R_{\alpha,s}$). *Let $s \geq 2$ be an integer and $\alpha \in K$. We have*

$$\Delta_1(R_{\alpha,1}(z)) = \frac{\alpha}{z(z+\alpha)} \quad \textit{and} \quad \Delta_1(R_{\alpha,s}(z)) = \frac{(-1)^s(s-1)!}{(z+\alpha)^s}. \tag{27}$$

Proof. According to Proposition 3.8, and since $\Delta_1 = \widehat{\mathfrak{M}}^{\mathrm{Inv}}(z)$, we have

$$\Delta_1(R_{\alpha,1}(z)) = \widehat{\mathfrak{M}}^{\mathrm{Inv}}(z)\Big(\mathfrak{M}^{\mathrm{Inv}}\left(\frac{(1+z)^{\alpha}-1}{z}\right)\Big) = \mathfrak{M}^{\mathrm{Inv}}\left((1+z)^{\alpha}-1\right) = \Delta_{\alpha}(\mathfrak{M}^{\mathrm{Inv}}(1)),$$

the last inequality coming from Lemma 5.3. Combined with $\mathfrak{M}^{\mathrm{Inv}}(1) = -1/z$, this yields the first equality of (27). We proceed in a similar way for the second equality. Note that since $R_{\alpha,s} = \tau_{\alpha}(R_s(z))$ and $\Delta_1 \circ \tau_{\alpha} = \tau_{\alpha} \circ \Delta_1$, we only have to prove the case $\alpha = 0$. Using Proposition 3.8, we obtain

$$\Delta_1(R_s(z)) = \widehat{\mathfrak{M}}^{\mathrm{Inv}}(z)\mathfrak{M}^{\mathrm{Inv}}\left(\frac{\log^{s-1}(1+z)}{z}\right) = \mathfrak{M}^{\mathrm{Inv}}\left(\log^{s-1}(1+z)\right) = \frac{d^{s-1}}{dz^{s-1}}\mathfrak{M}^{\mathrm{Inv}}(1),$$

the last equality coming from the first identity of Lemma 5.3. We conclude by using $\mathfrak{M}^{\mathrm{Inv}}(1) = -1/z$. □

6 Properties of the difference operator

We keep the notation of Section 4. Recall that for any $P(t) \in K[t]$, we denote by $[P]$ the operator "multiplication by $P(t)$". The difference operator $\boldsymbol{\Delta}_{-1} = \boldsymbol{\tau}_{-1} - [1]$ will be involved in the construction of the Padé approximants in Section 7. This motivates us to study its properties.

Lemma 6.1. *For any positive integer n and for any polynomial $P(t) \in K[t]$, we have*

$$\boldsymbol{\Delta}_{-1}^{n} \circ [P(t)] = \sum_{k=0}^{n} \binom{n}{k} \Big[\boldsymbol{\tau}_{-1}^{k} \circ \boldsymbol{\Delta}_{-1}^{n-k}(P(t))\Big] \circ \boldsymbol{\Delta}_{-1}^{k}.$$

Proof. Let $P(t), Q(t) \in K[t]$. We proceed by induction on n. For $n = 1$, we easily check that

$$\boldsymbol{\Delta}_{-1}\big(P(t)Q(t)\big) = \boldsymbol{\Delta}_{-1}(P(t)) \cdot Q(t) + \boldsymbol{\tau}_{-1}(P(t)) \cdot \boldsymbol{\Delta}_{-1}(Q(t)).$$

Suppose now that the lemma is true for a positive integer n. Then

$$\begin{aligned} \boldsymbol{\Delta}_{-1}^{n+1}\big(P(t)Q(t)\big) &= \boldsymbol{\Delta}_{-1}^{n}\Big(\boldsymbol{\Delta}_{-1}\big(P(t)Q(t)\big)\Big) \\ &= \boldsymbol{\Delta}_{-1}^{n}\Big(\boldsymbol{\Delta}_{-1}(P(t)) \cdot Q(t) + \boldsymbol{\tau}_{-1}(P(t)) \cdot \boldsymbol{\Delta}_{-1}(Q(t))\Big). \end{aligned}$$

We then get the result by applying the induction hypothesis with the pairs of polynomials $\big(\boldsymbol{\Delta}_{-1}(P(t)), Q(t)\big)$ and $\big(\boldsymbol{\tau}_{-1}(P(t)), \boldsymbol{\Delta}_{-1}(Q(t))\big)$ (and by using the commutativity of $\boldsymbol{\Delta}_{-1}$ and $\boldsymbol{\tau}_{-1}$). □

Lemma 6.2. *Let n be a positive integer. For any polynomial $P(t) \in K[t]$, we have*

$$[P(t)] \circ \boldsymbol{\Delta}_{-1}^{n} = \sum_{j=0}^{n} \binom{n}{j} \boldsymbol{\Delta}_{-1}^{n-j} \circ \Big[\boldsymbol{\tau}_{1}^{n-j} \circ \boldsymbol{\Delta}_{1}^{j}(P(t))\Big].$$

PROOF. We proceed by induction on n. For $n=1$, a direct computation ensures that for any polynomial $P(t)\in K[t]$, we have

$$[P(t)]\circ \boldsymbol{\Delta}_{-1} = \boldsymbol{\Delta}_{-1}\circ [\boldsymbol{\tau}_1(P(t))] + [\boldsymbol{\Delta}_1(P(t))]. \tag{28}$$

Let $n\geq 1$ be such that the assertion is true, and $P(t)\in K[t]$. By our induction hypothesis,

$$\begin{aligned}
[P(t)]\circ \boldsymbol{\Delta}_{-1}^{n+1} &= \sum_{j=0}^{n}\binom{n}{j}\boldsymbol{\Delta}_{-1}^{n-j}\circ\left[\boldsymbol{\tau}_1^{n-j}\circ\boldsymbol{\Delta}_1^{j}(P(t))\right]\circ\boldsymbol{\Delta}_{-1}\\
&= \sum_{j=0}^{n}\binom{n}{j}\boldsymbol{\Delta}_{-1}^{n-j}\circ\Big(\boldsymbol{\Delta}_{-1}\circ\left[\boldsymbol{\tau}_1^{n+1-j}\circ\boldsymbol{\Delta}_1^{j}(P(t))\right]+\left[\boldsymbol{\tau}_1^{n-j}\circ\boldsymbol{\Delta}_1^{j+1}(P(t))\right]\Big) \qquad (29)\\
&= \sum_{j=0}^{n+1}\binom{n+1}{j}\boldsymbol{\Delta}_{-1}^{n+1-j}\circ\left[\boldsymbol{\tau}_1^{n+1-j}\circ\boldsymbol{\Delta}_1^{j}(P(t))\right],
\end{aligned}$$

where we obtain (29) by applying (28) to the polynomial $\boldsymbol{\tau}_1^{n-j}\circ\boldsymbol{\Delta}_1^{j}(P(t))$, and by using the commutativity of $\boldsymbol{\Delta}_1$ and $\boldsymbol{\tau}_1$. This concludes our induction step. □

LEMMA 6.3. *Let $n, d, m_1,\dots,m_d$ be non-negative integers with $n,d\geq 1$, and $\alpha_1,\dots,\alpha_d\in K$. Set*

$$A(t)=\prod_{i=1}^{d}(t+\alpha_i)^{m_i+1} \quad \textit{and} \quad A_n(t)=\prod_{i=1}^{d}(t+\alpha_i)_n^{m_i+1}.$$

Suppose that $P(t)\in A_n(t)K[t]$. Then, we have the following properties.

(i) For each $k=0,\dots,n-1$, we have

$$\boldsymbol{\Delta}_{-1}^{k}(P(t))\in A(t)K[t].$$

(ii) For any polynomial $Q(t)\in K[t]$, we have

$$Q(t)\boldsymbol{\Delta}_{-1}^{n}(P(t))\in \boldsymbol{\Delta}_{-1}\big(A(t)K[t]\big)+P(t)\boldsymbol{\Delta}^{n}(Q(t)).$$

(iii) For any polynomial $Q(t)\in K[t]$ with $\deg Q(t)<n$, we have

$$Q(t)\boldsymbol{\Delta}_{-1}^{n}(P(t))\in \boldsymbol{\Delta}_{-1}\big(A(t)K[t]\big).$$

PROOF. (i). Recall the identity $\boldsymbol{\Delta}_{-1}^{k}=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\boldsymbol{\tau}_{-1}^{i}$. Write $P(t)=A_n(t)R(t)$, with $R(t)\in K[t]$, and fix an integer k with $0\leq k<n$. Then, we obtain

$$\boldsymbol{\Delta}_{-1}^{k}(P(t))=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}R(t-i)\prod_{j=1}^{d}(t+\alpha_j-i)_n^{m_j+1}. \tag{30}$$

We conclude by noticing that since $k<n$, for each $0\leq i\leq k$, the polynomial $\prod_{j=1}^{d}(t+\alpha_j-i)_n^{m_j+1}$ is divisible by $\prod_{j=1}^{d}(t+\alpha_j)^{m_j+1}=A(t)$.

(ii). Fix a polynomial $Q(t)\in K[t]$, and define

$$R_j(t)=P(t)\cdot\boldsymbol{\tau}_1^{n-j}\circ\boldsymbol{\Delta}_1^{j}\big(Q(t)\big)\in A_n(t)K[t]$$

for $j=0,\dots,n-1$. By Lemma 6.2, we have

$$Q(t)\boldsymbol{\Delta}_{-1}^{n}(P(t))=\boldsymbol{\Delta}_{-1}\left(\sum_{j=0}^{n-1}\binom{n}{j}\boldsymbol{\Delta}_{-1}^{n-1-j}\big(R_j(t)\big)\right)+P(t)\boldsymbol{\Delta}_1^{n}(Q(t)).$$

Applying (i) to the polynomial $R_j(t)$, we find $\boldsymbol{\Delta}_{-1}^{n-1-j}\big(R_j(t)\big)\in A(t)K[t]$ for $j=0,\dots,n-1$, hence (ii).

Finally, (iii) is a direct consequence of (ii), since $\boldsymbol{\Delta}_1^{n}(Q(t))=0$ as soon as $\deg Q(t)<n$. □

7 Construction of Padé approximants

We keep the notation of Sections 3 and 4. This section is devoted to the explicit construction of Padé-type approximants for the Laurent series $R_{\alpha,s}(z)$ introduced in Section 5. Let $d, m_1, \ldots, m_d$ be positive integers and $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \in K^d$ with $\alpha_1 = 0$. Denote by $\mathcal{S}$ the set of indices

$$\mathcal{S} = \{(i,s)\,;\, 1 \le i \le d \quad \text{and} \quad 1 \le s \le m_i + 1\} \setminus \{(1,1)\},$$

and set

$$M = \#\mathcal{S} = d - 1 + \sum_{i=1}^{d} m_i.$$

For simplicity, for any $\alpha \in \mathbb{Q}$ and any positive integer s, we write

$$\varphi_{\alpha,s} = \varphi_{R_{\alpha,s}}, \tag{31}$$

where $R_{\alpha,s}(z)$ as in Definition 5.1. We also introduce the following formal series of $K[[z]]$.

Definition 7.1. For any $(i,s) \in \mathcal{S}$, set

$$g_{i,s}(z) := \sum_{k=0}^{\infty} a_{i,s,k} z^k = \begin{cases} \dfrac{(1+z)^{\alpha_i} \log^{s-1}(1+z)}{z} & \text{if } s \ge 2 \\ \dfrac{(1+z)^{\alpha_i} - 1}{z} & \text{if } i \ge 2 \text{ and } s = 1. \end{cases}$$

If follows easily from that definition that $a_{1,s,0} = \cdots = a_{1,s,s-3} = 0$, and

$$a_{i,s,k} = \begin{cases} \displaystyle\sum_{\substack{\ell_i \ge 0 \\ \ell_1 + \cdots + \ell_{s-1} = k-s+2}} \frac{(-1)^{k-s+2}}{(\ell_1+1)\cdots(\ell_{s-1}+1)} & \text{if } i = 1 \text{ and } s \ge 2 \text{ and } k \ge s-2, \\ \displaystyle\sum_{j=0}^{k} \binom{\alpha_i}{j} a_{1,s,k-j} & \text{if } i \ge 2 \text{ and } s \ge 2, \\ \displaystyle\binom{\alpha_i}{k+1} & \text{if } i \ge 2 \text{ and } s = 1. \end{cases} \tag{32}$$

We will bound from above the absolute value of the coefficients $a_{i,s,k}$ at the end of the present section. Their p-adic absolute values are estimated in the proof of Lemma 9.5, while their denominators are studied in Lemma 9.7. We have the following key-properties.

Lemma 7.2. *For each $(i,s) \in \mathcal{S}$ and each integer $k \ge 0$, we have*

$$R_{\alpha_i,s}(z) = \mathfrak{M}^{\mathrm{Inv}}(g_{i,s}) \quad \textit{and} \quad \varphi_{\alpha_i,s}\left(\frac{(t)_k}{k!}\right) = (-1)^{k+1} a_{i,s,k}.$$

Proof. The first part is simply equivalent to the definition of $R_{\alpha_i,s}$. Then, we deduce the last part as a consequence of Corollary 4.9. □

Theorem 7.3. *Let ℓ, n be non-negative integers with $0 \le \ell \le M$. For any $(i,s) \in \mathcal{S}$, define the polynomials*

$$A_{n,\ell}(z) = A_\ell(z) = (-1)^\ell \frac{(z)_\ell}{\ell!} \prod_{j=1}^{d} \left((-1)^n \frac{(z+\alpha_j)_n}{n!}\right)^{m_j+1},$$

$$P_{n,\ell}(z) = P_\ell(z) = \Delta_{-1}^{n}\left(A_\ell(z)\right), \tag{33}$$

$$Q_{n,i,s,\ell}(z) = Q_{i,s,\ell}(z) = \varphi_{\alpha_i,s}\left(\frac{P_\ell(z) - P_\ell(t)}{z-t}\right). \tag{34}$$

(i) The vector of polynomials $\big(P_{n,\ell}(z), Q_{n,i,s,\ell}(z)\big)_{(i,s)\in\mathcal{S}}$ *forms a weight* $(n,\dots,n)\in\mathbb{N}^M$ *Padé-type approximant of* $\big(R_{\alpha_i,s}(z)\big)_{(i,s)\in\mathcal{S}}$.

(ii) We have the explicit formulas

$$P_{n,\ell}(z)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k+\ell}\frac{(z-k)_\ell}{\ell!}A_0(z-k)=\sum_{j=0}^{Mn+\ell}p_{n,j,\ell}\frac{(z)_j}{j!},$$

$$Q_{n,i,s,\ell}(z)=\sum_{j=1}^{Mn+\ell}p_{n,j,\ell}\left(\sum_{k=0}^{j-1}\frac{(-1)^{k+1}a_{i,s,k}\cdot k!}{(z)_{k+1}}\right)\frac{(z)_j}{j!},$$

where for each integer j *with* $0\le j\le Mn+\ell$, *the coefficient* $p_{n,j,\ell}$ *is given by*

$$p_{n,j,\ell}=p_{j,\ell}=\sum_{k=n}^{j+n}\binom{j+n}{k}(-1)^{n-k}\binom{k}{\ell}\prod_{r=1}^{d}\binom{k-\alpha_r}{n}^{m_r+1}.$$

(iii) For each $(i,s)\in\mathcal{S}$, *denote by*

$$\mathfrak{R}_{n,i,s,\ell}(z)=\mathfrak{R}_{i,s,\ell}(z)=P_{n,\ell}(z)R_{\alpha_i,s}(z)-Q_{n,i,s,\ell}(z)$$

the Padé approximation of $R_{\alpha_i,s}$. *Then*

$$\mathfrak{R}_{n,i,s,\ell}(z)=\sum_{k=n}^{\infty}\frac{k!}{(k-n)!}\frac{\varphi_{\alpha_i,s}\left((t+n)_{k-n}A_{n,\ell}(t)\right)}{z(z+1)\cdots(z+k)}.$$

Remark 7.4. In the case of $d=1$ and $\ell=0$, explicit Padé approximants of $R_2(z)$ have been studied by T. J. Stieltjes [45], J. Touchard [47] and L. Carlitz [10] (*confer* [35] and [7, Section 8]).

It is worth mentioning that our construction relies on the difference equation (27) satisfied by $R_{\alpha_i,s}(z)$. We will establish it through a difference analogue of the classical Rodrigues formula for orthogonal polynomials. In order to use Lemma 4.3, we will study the kernel of $\varphi_{\alpha_i,s}$. The methodology we present below investigates the Rodrigues formula for orthogonal polynomial systems, as discussed in [23], in the context of a difference equation.

Fix $(i,s)\in\mathcal{S}$. According to (27) the Laurent series $R_{\alpha_i,s}(z)$ satisfies the following difference equation:

$$(z+\alpha_i)^s\Delta_1(R_{\alpha_i,s}(z))=(-1)^s(s-1)! \qquad \text{if } s\ge 2,$$
$$z(z+\alpha_i)\Delta_1(R_{\alpha_i,1}(z))=\alpha_i \qquad \text{if } i\ge 2.$$

Applying Corollary 4.6 to $D=[(z+\alpha_i)^s]\circ\Delta_1$ and $D=[z(z+\alpha_i)]\circ\Delta_1$, we deduce from the above relation that

$$\boldsymbol{\Delta}_{-1}\big((t+\alpha_i)^sK[t]\big)\subseteq\ker\,\varphi_{\alpha_i,s} \qquad \text{if } s\ge 2, \tag{35}$$
$$\boldsymbol{\Delta}_{-1}\big(t(t+\alpha_i)K[t]\big)\subseteq\ker\,\varphi_{\alpha_i,1} \qquad \text{if } i\ge 2. \tag{36}$$

The exact kernel of $\varphi_{\alpha_i,s}$ will be determined in the next section, see Lemmas 8.4 and 8.5.

Lemma 7.5. *Let* n,ℓ *be integers with* $1\le n$ *and* $0\le\ell\le M$. *Then, for any* $(i,s)\in\mathcal{S}$, *we have*

$$t^kP_{n,\ell}(t)\in\ker\varphi_{\alpha_i,s} \qquad (0\le k\le n-1). \tag{37}$$

Proof. Lemma 6.3 (iii) implies that for $k = 0, \ldots, n-1$, we have

$$t^k P_{n,\ell}(t) = [t^k] \circ \boldsymbol{\Delta}_{-1}^n (A_{n,\ell}(t)) \in \boldsymbol{\Delta}_{-1} \circ [A(t)](K[t]),$$

where $A(t) = \prod_{r=1}^d (t+\alpha_r)^{m_r+1}$. Combining the above with (35) and (36) (and since $\alpha_1 = 0$) we deduce (37). □

Proof of Theorem 7.3 (i). The polynomial $A_{n,\ell}(z)$ has degree $n(M+1)+\ell$. The equality $\deg \Delta_{-1}(P) = \deg P - 1$ valid for any $P \in K[z]$, implies that $\deg P_{n,\ell}(z) = Mn + \ell$. We conclude by combining Lemma 7.5 and 4.3. □

Lemma 7.6. *For any polynomial $A(z) \in K[t]$ and any integer $n \geq 0$, we have*

$$A(z) = \sum_{j \geq 0} (-1)^j p_j \frac{(z)_j}{j!}, \tag{38}$$

$$\Delta_{-1}^n (A(z)) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} A(z-k) = \sum_{j\geq 0} (-1)^j p_{j+n} \frac{(z)_j}{j!}, \tag{39}$$

where

$$p_j = \mathrm{Eval}_{z=0} \circ \Delta_{-1}^j (A(z)) = \sum_{i=0}^j \binom{j}{i} (-1)^{j-i} A(-i). \tag{40}$$

Proof. The first equality of (39) and the second equality of (40) come from the identity $\Delta_{-1}^j = \sum_{i=0}^j \binom{j}{i} (-1)^{j-i} \tau_{-1}^i$. We deduce (38) with $p_j = \mathrm{Eval}_{z=0} \circ \Delta_{-1}^j (A(z))$ by using

$$\Delta_{-1} \left((-1)^j \frac{(z)_j}{j!} \right) = (-1)^{j-1} \frac{(z)_{j-1}}{(j-1)!} \quad (j \geq 1).$$

The last equality of (39) is obtained in a similar way. □

Lemma 7.7. *For each integer $j \geq 0$, we have*

$$\frac{(z)_j - (t)_j}{z-t} = (z)_j \sum_{k=0}^{j-1} \frac{(t)_k}{(z)_{k+1}}.$$

Proof. We proceed by induction on j. For $j = 0$, both sides are equal to 0. For the induction step, it suffices to use the identity

$$\frac{(z)_{j+1} - (t)_{j+1}}{z-t} = (z+j) \left(\frac{(z)_j - (t)_j}{z-t} \right) + (t)_j.$$

□

Proof of Theorem 7.3 (ii). By Lemma 7.6, we have

$$A_\ell(z) = \sum_{j=0}^{n(M+1)+\ell} a_{\ell,j} (-1)^j \frac{(z)_j}{j!}$$

where

$$a_{\ell,j} = \sum_{i=0}^j \binom{j}{i} (-1)^{j-i} A_\ell(-i) = \sum_{i=n}^j \binom{j}{i} (-1)^{j-i} \binom{i}{\ell} \prod_{k=1}^d \binom{i-\alpha_k}{n}^{m_k+1}, \tag{41}$$

where $A_\ell(-i) = 0$ for each $i < n$ (since $\alpha_1 = 0$). The expected formula for $A_\ell(z)$ follows. Lemma 7.6 together with (41) gives the explicit formula for $P_\ell(z) = \Delta_{-1}^n(A_\ell(z))$. Finally, using Lemma 7.7, we find

$$Q_{i,s,\ell} = \varphi_{\alpha_i,s}\left(\frac{P(z)-P(t)}{z-t}\right) = \sum_{j=0}^{Mn+\ell} \frac{p_{j,\ell}}{j!}\varphi_{\alpha_i,s}\left(\frac{(z)_j-(t)_j}{z-t}\right) = \sum_{j=1}^{Mn+\ell} \frac{p_{j,\ell}}{j!}(z)_j \sum_{k=0}^{j-1} \frac{k!}{(z)_{k+1}}\varphi_{\alpha_i,s}\left(\frac{(t)_k}{k!}\right),$$

and we conclude by Lemma 7.2. This completes the proof of Theorem 7.3 (ii). □

Proof of Theorem 7.3 (iii). Fix $(i,s) \in \mathcal{S}$ and for simplicity, set $f(z) = \mathfrak{R}_{i,s,\ell}(z)$. Write

$$f(z) = \sum_{k=n}^{\infty} \frac{r_{i,s,\ell,k}k!}{z(z+1)\cdots(z+k)}$$

(which is possible since according to Theorem 7.3 (i), the pair $\big(P_\ell(z), Q_{i,s,\ell}(z)\big)$ is a weight n Padé-type approximant of $\big(R_{\alpha_i,s}(z)\big)$. According to Lemma 4.8, we have $r_{i,s,\ell,k} = \varphi_f((t)_k/k!)$, so that it only remains to prove that

$$\varphi_f\left(\frac{(t)_k}{k!}\right) = \varphi_{\alpha_i,s}\left(\frac{(t+n)_{k-n}A_\ell(t)}{(k-n)!}\right) \tag{42}$$

for each $k \geq n$. Set $D = P_\ell(z)$. Then $D^* = P_\ell(t)$, and since $\mathfrak{R}_{i,s,\ell} = \pi(P_\ell R_{\alpha_i,s}) = \pi(D(R_{\alpha_i,s}))$, Proposition 4.5 yields

$$\varphi_f\left(\frac{(t)_k}{k!}\right) = \varphi_{\pi(D(R_{\alpha_i,s}))}\left(\frac{(t)_k}{k!}\right) = \varphi_{\alpha_i,s}\left(\frac{(t)_k}{k!}P_\ell(t)\right) = \varphi_{\alpha_i,s}\left(\left[\frac{(t)_k}{k!}\right]\circ \boldsymbol{\Delta}_{-1}^n(A_\ell(t))\right).$$

Using Lemma 6.3 (ii),

$$\left[\frac{(t)_k}{k!}\right]\circ \boldsymbol{\Delta}_{-1}^n(A_\ell(t)) \in \boldsymbol{\Delta}_{-1}\circ[A(t)](K[t]) + \left[\boldsymbol{\Delta}_1^n\left(\frac{(t)_k}{k!}\right)\right](A_\ell(t)),$$

where $A(t) = \prod_{j=1}^d (t+\alpha_j)^{m_j+1}$. By Eqs. (35) and (36), we have $\boldsymbol{\Delta}_{-1}\circ[A(t)](K[t]) \subseteq \ker\varphi_{\alpha_i,s}$. Finally, we deduce that

$$\varphi_f\left(\frac{(t)_k}{k!}\right) = \varphi_{\alpha_i,s}\left(\left[\boldsymbol{\Delta}_1^n\left(\frac{(t)_k}{k!}\right)\right](A_\ell(t))\right),$$

and we conclude by using the identity $\boldsymbol{\Delta}_1^n\left((t)_k/k!\right) = (t+n)_{k-n}/(k-n)!$. □

Absolute value of the coefficients $a_{i,s,k}$. We end this section by estimating roughly the absolute value of the coefficients $a_{i,s,k}$ appearing in Definition 7.1. In the proof of Lemma 9.5 we will estimate their p-adic absolute value. We start by estimating the binomial coefficients.

Lemma 7.8. *Let $k \geq 0$ be an integer and $\alpha \in \mathbb{C}$. We have*

$$\frac{|(\alpha)_k|}{k!} \leq \begin{cases} e^{|\alpha|-1}k^{|\alpha|-1} & \text{if } |\alpha| > 1 \text{ and } k > 0,\\ (k+1)^{-(1-|\alpha|)} & \text{if } |\alpha| \leq 1 \text{ or } k = 0.\end{cases}$$

In particular

$$\left|\binom{\alpha}{k}\right| \leq \frac{(|\alpha|)_k}{k!} \leq e^{|\alpha|}k^{|\alpha|}. \tag{43}$$

Proof. We may assume that $k > 0$ since $\binom{\alpha}{0} = 1$. Then, using the inequality $(1+x) \leq e^x$ valid for each $x \in [-1,\infty)$, we get

$$\left|\binom{\alpha}{k}\right| \leq \frac{(|\alpha|)_k}{k!} = \prod_{j=1}^{k}\Big(1+\frac{|\alpha|-1}{j}\Big) \leq \exp\Big(\sum_{j=1}^{k}\frac{|\alpha|-1}{j}\Big).$$

We obtain the expected upper bounds by combining the above with the estimates

$$\log(k+1) \leq \sum_{j=1}^{k} \frac{1}{j} \leq 1 + \log k,$$

and by distinguishing between the case $|\alpha| - 1 \leq 0$ and the case $|\alpha| - 1 > 0$. $\square$

LEMMA 7.9. *Let $(i, s) \in \mathcal{S}$ and $k \geq 0$ be an integer. Then*

$$|a_{i,s,k}| \leq e^{|\alpha_i|}(k+1)^{s+|\alpha_i|}.$$

PROOF. First, suppose $i = 1$. Then $s \geq 2$ and $\alpha_1 = 0$, and using (32) we obtain the crude estimate

$$|a_{1,s,k}| \leq \#\Big\{(\ell_1, \ldots, \ell_s) \in \mathbb{Z}_{\geq 0}^{s-1}\,;\, \ell_1 + \cdots + \ell_{s-1} = k - s + 2\Big\} \leq (k+1)^{s-1}. \tag{44}$$

We now assume that $i \geq 2$. If $s = 1$, then using (43) of Lemma 7.8 together with (32), we obtain

$$|a_{i,1,k}| = \left|\binom{\alpha_i}{k+1}\right| \leq e^{|\alpha_i|}(k+1)^{|\alpha_i|}. \tag{45}$$

If $s \geq 2$, we combine again (32) with (44) and (43) of Lemma 7.8 to get

$$|a_{i,s,k}| = \left|\sum_{j=0}^{k} \binom{\alpha_i}{j} a_{1,s,k-j}\right| \leq e^{|\alpha_i|}(k+1)^{s+|\alpha_i|}. \tag{46}$$

$\square$

8 Kernel of the formal integration maps

One of the crucial steps in proving our main Theorem 1.5 is to show that the Padé approximants constructed in Section 7 are linearly independent. In other words, we need the matrix, whose entries are formed by the Padé approximants, to be non-singular. This will be a consequence of the theorem below, which is the main result of this section.

We keep the notation of Section 4. Recall that the functions $R_{\alpha,s}$ (which are related to the polygamma functions) are introduced in Definition 5.1, and that the function $\Phi : f \mapsto \varphi_f$ is defined in Section 4.1.

DEFINITION 8.1. Given $\alpha \in K$ and an integer $s \geq 1$, we denote by $\varphi_{\alpha,s}$ the morphism $\Phi(R_{\alpha,s})$. In the case $\alpha = 0$ and $s \geq 2$, we simply write $\varphi_s = \Phi(R_s)$.

The following property will be useful when dealing with the functions $\varphi_{\alpha,s}$.

LEMMA 8.2. *Let $\alpha \in K$ and $s \geq 2$ an integer. Then*

$$\varphi_{\alpha,s} = \varphi_s \circ \boldsymbol{\tau}_{-\alpha}.$$

PROOF. This is a direct consequence of Corollary 4.7 (since by definition $R_{\alpha,s} = \tau_\alpha(R_s)$). $\square$

THEOREM 8.3. *Let $d \geq 1$ be an integer and $m_1, \ldots, m_d \geq 0$ be integers, with $m_1 \geq 1$. Set $M = m_1 + \cdots + m_d + d - 1$ and*

$$\mathcal{S} = \{(i, s)\,;\, 1 \leq i \leq d \quad \textit{and} \quad 1 \leq s \leq m_i + 1\} \setminus \{(1, 1)\}.$$

Fix non-negative integers $N_0, \ldots, N_d$. For $j = 1, \ldots, d$, let

$$\underline{r}(j) = \big(r_0^{(j)}, \ldots, r_{N_j}^{(j)}\big)$$

be a (N_j+1)*-tuple of integers with*

$$m_j + 1 \geq r_0^{(j)} \geq \cdots \geq r_{N_j}^{(j)} \geq 0, \tag{47}$$

and define

$$A(t) = \prod_{j=1}^{d} B_j(t+\alpha_j), \quad \textit{where } B_j(t) = \prod_{i=0}^{N_j} (t+i)^{r_i^{(j)}}.$$

Let $\alpha_1, \dots, \alpha_d \in K$ *satisfying the condition* $\alpha_1 = 0$ *and*

$$\alpha_i - \alpha_j \notin \mathbb{Z} \textit{ for any distinct indices } i, j \in \{1, \dots, d\}. \tag{48}$$

Then, the following $M \times M$ *matrix is non-singular*

$$\mathcal{M} := \Big(\varphi_{\alpha_i,s}(t^\ell A(t))\Big)_{\substack{(i,s)\in\mathcal{S}\\ 0\leq \ell < M}} = \begin{pmatrix} \varphi_{\alpha_1,2}\big(A(t)\big) & \varphi_{\alpha_1,2}\big(tA(t)\big) & \cdots & \varphi_{\alpha_1,2}\big(t^{M-1}A(t)\big) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{\alpha_1,m_1+1}\big(A(t)\big) & \varphi_{\alpha_1,m_1+1}\big(tA(t)\big) & \cdots & \varphi_{\alpha_1,m_1+1}\big(t^{M-1}A(t)\big) \\ \varphi_{\alpha_2,1}\big(A(t)\big) & \varphi_{\alpha_2,1}\big(tA(t)\big) & \cdots & \varphi_{\alpha_2,1}\big(t^{M-1}A(t)\big) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{\alpha_2,m_2+1}\big(A(t)\big) & \varphi_{\alpha_2,m_2+1}\big(tA(t)\big) & \cdots & \varphi_{\alpha_2,m_2+1}\big(t^{M-1}A(t)\big) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{\alpha_d,1}\big(A(t)\big) & \varphi_{\alpha_d,1}\big(tA(t)\big) & \cdots & \varphi_{\alpha_d,1}\big(t^{M-1}A(t)\big) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{\alpha_d,m_d+1}\big(A(t)\big) & \varphi_{\alpha_d,m_d+1}\big(tA(t)\big) & \cdots & \varphi_{\alpha_d,m_d+1}\big(t^{M-1}A(t)\big) \end{pmatrix}.$$

The strategy of our proof is as follows. We easily show that any point in the kernel of $\mathcal{M}$ gives rise to a polynomial $Q(t) \in K[t]$ of degree $\leq M-1$ satisfying

$$Q(t)A(t) \in \bigcap_{(i,s)\in\mathcal{S}} \ker \varphi_{\alpha_i,s}.$$

The core of the demonstration of the theorem consists in expressing the above subspace as the image by the operator $\boldsymbol{\Delta}_{-1}$ of a rather simple ideal of $K[t]$ (see Section 8.1). This will allow us in Section 8.2 to prove that a non-zero polynomial $Q(t)$ as above has degree at least M, hence a contradiction if $Q(t) \neq 0$.

8.1 Study of the kernels

We keep the notation of Section 4.1 for the operators $\boldsymbol{\Delta}_\alpha$, τ_α and the linear maps $\varphi_{\alpha,s} = \Phi(R_{\alpha,s})$. We start by expressing the kernel of the linear maps $\varphi_{\alpha,s}$ in a simple way.

Lemma 8.4. *Let* $s \geq 2$ *be an integer and fix a shift* $\alpha \in K$*.*

(i) We have

$$\varphi_{\alpha,s} \circ \boldsymbol{\Delta}_{-1} = (-1)^s \mathrm{Eval}_{t=-\alpha} \circ \left(\frac{d}{dt}\right)^{s-1}.$$

(ii) The kernel of $\varphi_{\alpha,s}$ *is*

$$\ker \varphi_{\alpha,s} = \boldsymbol{\Delta}_{-1} \circ \boldsymbol{\tau}_\alpha \left(\left\langle t, \dots, t^{s-2}, t^s, t^{s+1}, \dots \right\rangle_K\right). \tag{49}$$

(iii) For any non-negative integer m, we have

$$\bigcap_{s=2}^{m+1} \ker \varphi_{\alpha,s} = \boldsymbol{\Delta}_{-1}\left((t+\alpha)^{m+1}K[t]\right). \tag{50}$$

with the convention that the left-hand side is equal to $K[t]$ if $m=0$.

PROOF. Since by Lemma 8.2 we have $\varphi_{\alpha,s} = \varphi_s \circ \boldsymbol{\tau}_{-\alpha}$, and since $\boldsymbol{\Delta}_{-1}$ commutes with any shift operator, it suffices to prove the lemma when $\alpha = 0$. Proposition 4.5 gives $\varphi_s \circ \boldsymbol{\Delta}_{-1} = \varphi_{\Delta_1(R_s)}$. Eq. (27) of Proposition 5.4 and a direct computation yield

$$\varphi_{\Delta_1(R_s)} = (-1)^s (s-1)! \varphi_{1/z^s} = (-1)^s \mathrm{Eval}_{t=0} \circ \left(\frac{d}{dt}\right)^{s-1}.$$

Hence (i). We deduce that

$$H := \boldsymbol{\Delta}_{-1}\left(\ker \mathrm{Eval}_{t=0} \circ \left(\frac{d}{dt}\right)^{s-1}\right) = \boldsymbol{\Delta}_{-1}\left(\langle t, \ldots, t^{s-2}, t^s, t^{s+1}, \ldots \rangle_K\right) \subseteq \ker \varphi_{0,s}.$$

Since H is an hyperplane of $K[t]$ and φ_s is a non-zero linear form, the above inclusion is an equality, and (49) follows. Eq. (50) is a consequence of (49). If $m = 0$, we simply have $\boldsymbol{\Delta}_{-1}\big((t+\alpha)K[t]\big) = K[t]$. ☐

LEMMA 8.5. *Fix a shift $\alpha \in K \setminus \{0\}$.*

(i) We have

$$\varphi_{\alpha,1} \circ \boldsymbol{\Delta}_{-1} = \mathrm{Eval}_{t=0} - \mathrm{Eval}_{t=-\alpha}. \tag{51}$$

(ii) The kernel of $\varphi_{\alpha,1}$ is

$$\ker \varphi_{\alpha,1} = \boldsymbol{\Delta}_{-1}\Big(t(t+\alpha)K[t]\Big)$$

REMARK 8.6. Note that $\varphi_{0,1} = 0$, so that $\ker \varphi_{0,1} = K[t]$.

PROOF. (i). Proposition 4.5 yields $\varphi_{\alpha,1} \circ \boldsymbol{\Delta}_{-1} = \varphi_{\Delta_1(R_{\alpha,1})}$. Combining this with Eq. (27) of Proposition 5.4, we deduce that $\varphi_{\Delta_1(R_{\alpha,1})} = \varphi_f$, where

$$f(z) = \frac{\alpha}{z(z+\alpha)} = \frac{1}{z} - \frac{1}{z+\alpha}.$$

We conclude by noting that $\varphi_{1/z} = \mathrm{Eval}_{t=0}$, and $\varphi_{1/(z+\alpha)} = \varphi_{1/z} \circ \boldsymbol{\tau}_{-\alpha}$ by Corollary 4.7.

(ii). Eq. (51) easily implies that

$$H := \boldsymbol{\Delta}_{-1}\big(t(t+\alpha)K[t]\big) \subseteq \ker \varphi_{\alpha,1}$$

(note that this is also a consequence of (36)). Since H is a hyperplane of $K[t]$ and $\varphi_{\alpha,1} \neq 0$, the above inclusion is an equality. ☐

We now establish a generalization of Eq. (50) of Lemma 8.4 by taking into account several shifts simultaneously.

LEMMA 8.7. *Let $d, m_1, \ldots, m_d$ be non-negative integers with $d \geq 1$. Let $\alpha_1, \ldots, \alpha_d$ be pairwise distinct elements of K, with $\alpha_1 = 0$. We have*

$$\bigcap_{i=1}^{d}\left(\bigcap_{s=1}^{m_i+1} \ker \varphi_{\alpha_i,s}\right) = \boldsymbol{\Delta}_{-1}\Big(t^{m_1+1}(t+\alpha_2)^{m_2+1}\cdots(t+\alpha_d)^{m_d+1}K[t]\Big).$$

Proof. By Lemma 8.4 (iii) and Lemma 8.5 (ii), we have to prove that the two following subspaces

$$V := \bigcap_{i=1}^{d} \mathbf{\Delta}_{-1}\left((t+\alpha_i)^{m_i+1}K[t]\right) \bigcap_{i=2}^{d} \mathbf{\Delta}_{-1}\Big(t(t+\alpha_i)K[t]\Big),$$

$$W := \mathbf{\Delta}_{-1}\Big(\prod_{i=1}^{d}(t+\alpha_i)^{m_i+1}K[t]\Big),$$

are equal. The inclusion $W \subseteq V$ is trivial. Now, fix $P(t) \in V$, and let us prove that $P(t) \in W$. Since $P(t) \in \mathbf{\Delta}_{-1}\left(t^{m_1+1}K[t]\right)$, there exists $Q(t) \in K[t]$ such that

$$P(t) = \mathbf{\Delta}_{-1}\Big(t^{m_1+1}Q(t)\Big).$$

To conclude, it suffices to prove that $\prod_{i=2}^{d}(t+\alpha_i)^{m_i+1}$ divides $Q(t)$. Given $i \in \{2, \ldots, d\}$, there exist $R_i(t), S_i(t) \in K[t]$ such that

$$P(t) = \mathbf{\Delta}_{-1}\Big((t+\alpha_i)^{m_i+1}R_i(t)\Big) = \mathbf{\Delta}_{-1}\Big(t(t+\alpha_i)S_i(t)\Big).$$

Since $\ker \mathbf{\Delta}_{-1} = K \subseteq K[t]$, we deduce the existence of $a_i, b_i \in K$ satisfying

$$t(t+\alpha_i)S_i(t) = (t+\alpha_i)^{m_i+1}R_i(t) + a_i = t^{m_1+1}Q(t) + b_i.$$

Evaluating at $t = -\alpha_i$ and $t = 0$, we find $a_i = b_i = 0$, so that

$$t^{m_1+1}Q(t) = (t+\alpha_i)^{m_i+1}R_i(t).$$

As $\alpha_i \neq 0$, it follows that $(t+\alpha_i)^{m_i+1}$ divides $Q(t)$. Since the α_i's are all distinct, we deduce that the polynomial $\prod_{i=2}^{d}(t+\alpha_i)^{m_i+1}$ divides $Q(t)$. Hence $P(t) \in W$. □

8.2 Proof of Theorem 8.3

Lemma 8.8. *Fix two non-negative integers N, m. Let $\underline{r} = (r_0, \ldots, r_N)$ be a $(N+1)$-tuple of integers satisfying $m+1 \geq r_0 \geq \cdots \geq r_N \geq 0$, and define*

$$B_{\underline{r}}(t) = \prod_{i=0}^{N}(t+i)^{r_i}.$$

(i) Let $Q(t), R(t) \in K[t]$ be two polynomials satisfying $Q(t)B_{\underline{r}}(t) = \mathbf{\Delta}_{-1}\big(t^{m+1}R(t)\big)$. Then the polynomial $B_{\underline{r}}(t+1)$ divides $R(t)$.

(ii) We have

$$B_{\underline{r}}(t)K[t] \cap \mathbf{\Delta}_{-1}\Big(t^{m+1}K[t]\Big) = \mathbf{\Delta}_{-1}\Big(t^{m+1}B_{\underline{r}}(t+1)K[t]\Big).$$

Proof. For simplicity, write $A(t) = B_{\underline{r}}(t)$. We may assume that $Q(t)$ and $R(t)$ are non-zero, otherwise (i) is automatic. By induction on k, let us prove that $(t+k+1)^{r_k}$ is a factor of $R(t)$ for $i = 0, \ldots, N$. By hypothesis, we have

$$Q(t)A(t) = \mathbf{\Delta}_{-1}(t^{m+1}R(t)) = (t-1)^{m+1}R(t-1) - t^{m+1}R(t). \tag{52}$$

Since t^{r_0} divides $A(t)$ and $t^{m+1}R(t)$ (since $r_0 \leq m+1$), necessarily t^{r_0} also divides $(t-1)^{m+1}R(t-1)$. It follows that t^{r_0} divides $R(t-1)$, or equivalently $(t+1)^{r_0}$ divides $R(t)$. Suppose now that $(t+k+1)^{r_k}$ divides $R(t)$ for some integer k with $0 \leq k < N$. Then, $(t+k+1)^{r_{k+1}}$ divides $A(t)$ as well as $R(t)$, since $r_{k+1} \leq r_k$.

Eq. (52) ensures that $(t+k+1)^{r_{k+1}}$ divides $(t-1)^{m+1}R(t-1)$. We deduce that $(t+k+2)^{r_{k+1}}$ divides $R(t)$, which concludes our induction step. Therefore, the polynomial

$$\prod_{k=0}^{N}(t+k+1)^{r_k} = A(t+1)$$

divides $R(t)$, hence (i). It follows that

$$B_{\underline{r}}(t)K[t] \cap \mathbf{\Delta}_{-1}\Big(t^{m+1}K[t]\Big) \subseteq \mathbf{\Delta}_{-1}\Big(t^{m+1}B_{\underline{r}}(t+1)K[t]\Big). \tag{53}$$

Conversely, the hypothesis $m+1 \geq r_0 \geq \cdots \geq r_N \geq 0$ implies that $B_{\underline{r}}(t)$ divides $t^{m+1}B_{\underline{r}}(t+1)$. Thus $B_{\underline{r}}(t)$ also divides $\mathbf{\Delta}_{-1}\big(t^{m+1}B_{\underline{r}}(t+1)\big)$. We easily deduce that (53) is an equality, hence (ii). □

We now establish a generalization of Lemma 8.8 which will be needed in order to prove Theorem 8.3.

LEMMA 8.9. *We keep the notation of Theorem 8.3 and put*

$$B(t) = \prod_{i=0}^{d}(t+\alpha_i)^{m_i+1}.$$

(i) Let $Q(t), R(t) \in K[t]$ be two polynomials satisfying $Q(t)A(t) = \mathbf{\Delta}_{-1}\big(B(t)R(t)\big)$. Then the polynomial $A(t+1)$ divides $R(t)$.

(ii) We have

$$A(t)K[t] \cap \mathbf{\Delta}_{-1}\Big(B(t)K[t]\Big) = \mathbf{\Delta}_{-1}\Big(B(t)A(t+1)K[t]\Big).$$

PROOF. For $j = 1, \ldots, d$, write

$$Q(t)A(t) = Q_j(t+\alpha_j)B_j(t+\alpha_j) \quad \text{and} \quad B(t)R(t) = (t+\alpha_j)^{m_j+1}R_j(t+\alpha_j),$$

with $Q_j(t), R_j(t) \in K[t]$. By hypothesis, we have $Q_j(t)B_j(t) = \mathbf{\Delta}_{-1}\big(t^{m_j+1}R_j(t)\big)$. Lemma 8.8 (i) implies that $B_j(t+1)$ divides $R_j(t)$. Equivalently, $B_j(t+\alpha_j+1)$ divides

$$R_j(t+\alpha_j) = R(t)\prod_{\substack{i=1\\ i\neq j}}^{d}(t+\alpha_i)^{m_i+1}.$$

Our hypothesis (48) on the α_i ensures that $B_j(t+\alpha_j+1)$ and $\displaystyle\prod_{i=1,i\neq j}^{d}(t+\alpha_i)^{m_i+1}$ are coprime polynomials. So $B_j(t+\alpha_j+1)$ divides $R(t)$. Furthermore, (48) also implies that $B_1(t+\alpha_1+1), \ldots, B_d(t+\alpha_d+1)$ are coprime. We conclude that the product $B_1(t+\alpha_1+1)\cdots B_d(t+\alpha_d+1) = A(t+1)$ divides $R(t)$. Hence (i). We also deduce that

$$A(t)K[t] \cap \mathbf{\Delta}_{-1}\Big(B(t)K[t]\Big) \subseteq \mathbf{\Delta}_{-1}\Big(B(t)A(t+1)K[t]\Big).$$

Conversely, the hypothesis (47) implies that $A(t)$ divides $B(t)A(t+1)$. Thus $A(t)$ also divides $\mathbf{\Delta}_{-1}\big(B(t)(t+1)\big)$ and the above inclusion is an equality. □

Proof of Theorem 8.3. Given $a_0, \ldots, a_{M-1} \in K$, we have

$$\mathcal{M}C = \begin{pmatrix} \varphi_{1,2}\Big(Q(t)A(t)\Big) \\ \vdots \\ \varphi_{d,m_d+1}\Big(Q(t)A(t)\Big) \end{pmatrix}, \qquad \text{where } C = \begin{pmatrix} a_0 \\ \vdots \\ a_{M-1} \end{pmatrix} \quad \text{and} \quad Q(t) = \sum_{k=0}^{M-1} a_k t^k.$$

Suppose that $C \in \ker \mathcal{M}$. Then, writing $B(t) = \prod_{i=1}^{d}(t+\alpha_i)^{m_i+1}$, we have

$$Q(t)A(t) \in \bigcap_{(i,s)\in\mathcal{S}} \ker \varphi_{\alpha_i,s} = \mathbf{\Delta}_{-1}\Big(B(t)K[t]\Big),$$

the last equality coming from Lemma 8.7. Using Lemma 8.9 (ii), we deduce that there exists $R(t) \in K[t]$ such that

$$Q(t)A(t) = \mathbf{\Delta}_{-1}\big(B(t)A(t+1)R(t)\big).$$

We find

$$\begin{aligned} M-1+\deg A(t) \geq \deg Q(t) + \deg A(t) &= \deg \mathbf{\Delta}_{-1}\Big(B(t)A(t+1)R(t)\Big) \\ &= \deg B(t) + \deg A(t) + \deg R(t) - 1. \end{aligned}$$

Since $\deg B(t) = M+1$, it follows that $\deg R(t) \leq -1$, hence $R(t) = 0$. As a consequence $Q(t) = 0$, or equivalently, $C = 0$. Thus $\ker \mathcal{M} = \{0\}$. □

8.3 Linear independence of the Padé approximants

We keep the notation of Section 7, with $K = \mathbb{Q}$. Let ℓ, n be non-negative integers with $0 \leq \ell \leq M$. For each $(i,s) \in \mathcal{S}$, the polynomials $P_{n,\ell}(z)$, $Q_{n,i,s,\ell}(z)$, and the Padé approximation

$$\mathfrak{R}_{n,i,s,\ell}(z) = P_{n,\ell}(z)R_{\alpha_i,s}(z) - Q_{n,i,s,\ell}(z)$$

of $R_{\alpha_i,s}(z)$ are defined in Theorem 7.3. The main result of this subsection is Theorem 8.10 below, which ensures the crucial non-vanishing property of certain determinants associated with the above Padé approximants. It uses the following notation. For $\ell = 0, \ldots, M$, define the column vectors

$$\begin{aligned} \boldsymbol{p}_{n,\ell}(z) &= {}^t\Big(P_{n,\ell}(z), Q_{n,1,2,\ell}(z), \ldots, Q_{n,1,m_1+1,\ell}(z), \ldots, Q_{n,d,1,\ell}(z), \cdots, Q_{n,d,m_d+1,\ell}(z)\Big) \\ &= {}^t\Big(P_{n,\ell}(z), Q_{n,i,s,\ell}(z)\Big)_{(i,s)\in\mathcal{S}}, \end{aligned}$$

and form the $(M+1)\times(M+1)$ matrix

$$\mathcal{M}_n(z) = \big(\boldsymbol{p}_{n,0}(z), \ldots, \boldsymbol{p}_{n,M}(z)\big).$$

THEOREM 8.10. *We have* $\det \mathcal{M}_n(z) \in \mathbb{Q}^{\times}$. *In particular, for any* $x \in \mathbb{Q}$, *the Padé approximants* $\boldsymbol{p}_{n,0}(x), \ldots, \boldsymbol{p}_{n,M}(x)$ *are linearly independent over* K.

Theorem 8.10 is a direct consequence of Lemma 8.11 and Proposition 8.12 below. Our strategy is the following. By definition $\mathcal{M}_n(z)$ is a polynomial. We show in Lemma 8.11, which is essentially an application of [15, Lemma 4.2 (ii)], that this polynomial is a constant, and we reduce the problem to showing that another determinant Θ_n is non-zero. This last property, established in Proposition 8.12, will be a consequence of Theorem 8.3. In order to prove the above results, let us introduce more notation. Define

$$\mathfrak{D}_n(z) = \det \mathcal{M}_n(z) \quad \text{and} \quad \Theta_n = \det(\varphi_{\alpha_i,s}(A_{n,\ell}(t)))_{\substack{0\leq \ell\leq M-1 \\ (i,s)\in\mathcal{S}}} \in \mathbb{Q},$$

where

$$A_{n,\ell}(t) = (-1)^{\ell}\frac{(t)_{\ell}}{\ell!}\prod_{i=1}^{d}\left((-1)^n\frac{(t+\alpha_i)_n}{n!}\right)^{m_i+1}$$

is defined in Theorem 7.3. For $\ell = 0, \dots, M$, denote by $\mathbf{r}_{n,\ell}(z)$ the column vector

$$\begin{aligned}\mathbf{r}_{n,\ell}(z) &= {}^t\Big(P_{n,\ell}(z), \mathfrak{R}_{n,1,2,\ell}(z), \dots, \mathfrak{R}_{n,1,m_1+1,\ell}(z), \dots, \mathfrak{R}_{n,d,1,\ell}(z), \cdots, \mathfrak{R}_{n,d,m_d+1,\ell}(z)\Big) \\ &= {}^t\Big(P_{n,\ell}(z), \mathfrak{R}_{n,i,s,\ell}(z)\Big)_{(i,s)\in\mathcal{S}},\end{aligned}$$

and form the $(M+1) \times (M+1)$ matrix

$$\mathcal{M}_{n,\mathfrak{R}}(z) = \big(\mathbf{r}_{n,0}(z), \dots, \mathbf{r}_{n,M}(z)\big).$$

Then, by definition of the Padé approximations $\mathfrak{R}_{i,s,\ell}(z)$, we have

$$U(z)\mathcal{M}_n(z) = \mathcal{M}_{n,\mathfrak{R}}(z), \quad \text{where } U(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ R_{n,1,2}(z) & -1 & 0 & \vdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots & \cdots & \cdots & \vdots \\ R_{n,1,m_1+1}(z) & 0 & \cdots & -1 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ R_{n,d,1}(z) & 0 & 0 & \vdots & -1 & \vdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ R_{n,d,m_d+1}(z) & 0 & \cdots & 0 & \ddots & \cdots & -1 \end{pmatrix}. \tag{54}$$

Lemma 8.11. *There exists $c \in \mathbb{Q}^\times$ such that $\mathfrak{D}_n(z) = c \cdot \Theta_n$.*

Proof. The entries of the first row of $\det \mathcal{M}_{n,\mathfrak{R}}(z)$ are the polynomials $P_{n,0}(z), P_{n,1}(z), \dots, P_{n,M}(z)$, which have degrees $nM, nM+1, \dots, nM+M$ respectively. Thus

$$P_{n,0}(z), \dots, P_{n,M-1}(z) \in \mathbb{Q}[z]_{\le (n+1)M-1} \quad \text{and} \quad P_{n,M}(z) \in \widetilde{c}z^{(n+1)M} + \mathbb{Q}[z]_{\le (n+1)M-1}, \tag{55}$$

where $\widetilde{c} \in \mathbb{Q}^\times$ denotes the leading coefficient of the polynomial $P_{n,M}(z)$. On the other hand, for each $(i,s) \in \mathcal{S}$ and each ℓ with $0 \le \ell \le M$, Theorem 7.3 (iii) ensures that

$$\mathfrak{R}_{n,i,s,\ell} \in \frac{n!\varphi_{\alpha_i,s}(A_{n,\ell}(t))}{z^{n+1}} + \frac{1}{z^{n+2}}\mathbb{Q}[[1/z]]. \tag{56}$$

For the sake of completion, we now recall the main arguments of [15, Lemma 4.2 (ii)]. Expanding $\det \mathcal{M}_{n,\mathfrak{R}}(z)$ along its first row and using (55) together with (56), we find

$$\det \mathcal{M}_{n,\mathfrak{R}}(z) \in (-1)^M (n!)^M \widetilde{c} \det\big(\varphi_{\alpha_i,s}(A_{n,\ell}(t))\big)_{\substack{0\le\ell\le M-1 \\ (i,s)\in\mathcal{S}}} + \frac{1}{z}\mathbb{Q}[[1/z]].$$

Finally, according to (54), $\det \mathcal{M}_{n,\mathfrak{R}}(z) = (-1)^M \det \mathcal{M}_n(z) \in \mathbb{Q}[z]$. Combined with the above, we conclude that $\det \mathcal{M}_{n,\mathfrak{R}}(z) = (-1)^M (n!)^M \widetilde{c}\,\Theta_n$. □

Proposition 8.12. *Let $\mathbf{n} = (n_{1,0}, \dots, n_{1,m_1}, \dots, n_{d,0}, \dots, n_{d,m_d})$ be a $(M+1)$-tuple of non-negative integers. Then*

$$\Theta_{\mathbf{n}} := \det\left(\varphi_{\alpha_i,s}\left((-1)^\ell \frac{(t)_\ell}{\ell!} \prod_{k=1}^{d} \prod_{j=0}^{m_k} (-1)^{n_{k,j}} \frac{(t+\alpha_k)_{n_{k,j}}}{n_{k,j}!}\right)\right)_{\substack{0\le\ell\le M-1 \\ (i,s)\in\mathcal{S}}} \ne 0.$$

In particular $\Theta_n = \Theta_{(n,\dots,n)} \ne 0$.

Proof. Fix non-negative integers $m, n_0, \dots, n_m$, and set $N = \max\{n_0, \dots, n_m\}$. For $k = 0, \dots, N$, define r_k as the number of indices $i \in \{0, \dots, m\}$ such that $n_i > k$, and set $\underline{r} = (r_0, \dots, r_N)$. Then $m + 1 \geq r_0 \geq \cdots \geq r_N = 0$ and

$$\prod_{i=0}^{m} (t)_{n_i} = B_{\underline{r}}(t) = \prod_{i=0}^{N} (t + i)^{r_i},$$

where $B_{\underline{r}}(t)$ is as in Lemma 8.8. We conclude that the polynomial

$$\prod_{k=1}^{d} \prod_{j=0}^{m_k} (t + \alpha_k)_{n_{k,j}}$$

has the same form as the polynomial $A(t)$ in the statement of Theorem 8.3, hence

$$0 \neq \det \Big(\varphi_{\alpha_i,s}\Big(t^{\ell} A(t)\Big) \Big)_{\substack{0 \leq \ell \leq M-1 \\ (i,s) \in \mathcal{S}}} = \pm \Theta_{\mathbf{n}} \cdot \left(\prod_{\ell=0}^{M-1} \ell! \right) \cdot \left(\prod_{j=0}^{m_k} n_{k,j}! \right)^{M}.$$

We conclude that $\Theta_{\mathbf{n}} \neq 0$. □

Although we will not need it in the following, it seems that in the simpler case $d = 1$, we can express the determinant $\Theta_{\mathbf{n}}$ in a simple way.

Conjecture 8.13. Let $m, n_0, \dots, n_m$ non-negative integers with $m \geq 1$. The following identity holds

$$\det \left(\varphi_s \left(\frac{(t)_\ell}{\ell!} \prod_{j=0}^{m} \frac{(t)_{n_j}}{n_j!} \right) \right)_{\substack{0 \leq \ell \leq m-1 \\ 2 \leq s \leq m+1}} = \frac{(-1)^{m(m+1)/2} m! \prod_{i=0}^{m} n_i!}{(n_0 + \cdots + n_m + m)!}.$$

9 Estimates

We keep the notation of Section 7, with $K = \mathbb{Q}$. So

$$\mathcal{S} = \{(i, s)\, ;\, 1 \leq i \leq d \quad \text{and} \quad 1 \leq s \leq m_i + 1\} \setminus \{(1, 1)\},$$

and for each $\alpha \in \mathbb{Q}$ and any positive integer s, we have

$$\varphi_{\alpha,s} = \varphi_{R_{\alpha,s}},$$

(see Definitions 5.1 and 8.1). Let ℓ, n be non-negative integers with $0 \leq \ell \leq M$. For each $(i, s) \in \mathcal{S}$, the polynomials $P_{n,\ell}(z)$, $Q_{n,i,s,\ell}(z)$, and the Padé approximation

$$\mathfrak{R}_{n,i,s,\ell}(z) = P_{n,\ell}(z) R_{\alpha_i,s}(z) - Q_{n,i,s,\ell}(z)$$

of $R_{\alpha_i,s}(z)$ are defined in Theorem 7.3. Recall that the polynomial $P_{n,\ell}(z)$ has degree $nM + \ell$.

9.1 Absolute value of the Padé approximants

We keep the notation introduced at the beginning of Section 9. We describe the asymptotic behavior, as n goes to infinity, of the polynomials $P_{n,\ell}(z)$ and $Q_{n,i,s,\ell}(z)$ evaluated at a fixed rational number x. In Section 10 we will explain how we can improve the rough estimate (58) below thanks to Perron's second theorem.

PROPOSITION 9.1. *Let $x \in \mathbb{Q}$. Then, for any $(i,s) \in \mathcal{S}$ and any integer ℓ with $0 \le \ell \le M$, we have*

$$\limsup_{n\to\infty} |P_{n,\ell}(x)|^{1/n} \le 1, \tag{57}$$

$$\limsup_{n\to\infty} |Q_{n,i,s,\ell}(x)|^{1/n} \le \rho(M) := \left(2\frac{(M+1)^{M+1}}{M^M}\right)^{M+1}. \tag{58}$$

PROOF. Fix $(i,s) \in \mathcal{S}$ and integer ℓ with $0 \le \ell \le M$. Let n be a positive integer. We first prove (57). Theorem 7.3 (ii) together with the identity $(y-k)_n = (y-k)_k(y)_{n-k}$ valid for each $k \le n$ yields

$$\begin{aligned} P_{n,\ell}(z) &= \pm \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(z-k)_\ell}{\ell!} \prod_{j=1}^{d} \left(\frac{(z+\alpha_j-k)_k (z+\alpha_j)_{n-k}}{n!} \right)^{m_j+1} \\ &= \pm \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(z-k)_\ell}{\ell!} \prod_{j=1}^{d} \left(\frac{(z+\alpha_j-k)_k}{k!} \frac{(z+\alpha_j)_{n-k}}{(n-k)!} \binom{n}{k}^{-1} \right)^{m_j+1}. \end{aligned}$$

Noticing that $|(x+\alpha_j-k)_k| \le (1+|x+\alpha_j|)_k$ and using Lemma 7.8, we find for $k=0,\dots,n$ the upper bounds

$$\max\left\{ \frac{|(x+\alpha_j-k)_k|}{k!}, \frac{|(x+\alpha_j)_{n-k}|}{(n-k)!} \right\} \le (en)^{|x+\alpha_j|}.$$

We deduce the rough estimate

$$|P_{n,\ell}(x)| \le \frac{(|x|+n)^\ell}{\ell!} (en)^\beta = e^{o(n)}, \quad \text{where } \beta = 2\sum_{j=1}^{d} |x+\alpha_j|(m_j+1),$$

hence (57). We now prove (58). By Theorem 7.3 (ii), we have

$$Q_{i,s,\ell}(z) = \sum_{j=1}^{Mn+\ell} p_{n,j,\ell} \cdot b_{j,i,s}(z),$$

where

$$b_{j,i,s}(z) := \sum_{k=0}^{j-1} (-1)^{k+1} a_{i,s,k} \frac{(z+k+1)\cdots(z+j-1)}{(k+1)\cdots(j-1)j}$$

(the coefficients $a_{i,s,k}$ are as in Definition 7.1), and

$$p_{n,j,\ell} = \sum_{k=n}^{j+n} \binom{j+n}{k} (-1)^{n-k} \binom{k}{\ell} \prod_{r=1}^{d} \binom{k-\alpha_r}{n}^{m_r+1} \tag{59}$$

as in Theorem 7.3 (ii). Evaluating at $z=x$ and using Lemma 7.8, we find for $j=1,\dots,Mn+\ell$

$$\frac{|(x+k+1)\cdots(x+j-1)|}{(k+1)\cdots(j-1)} \le \frac{(|x|+k+1)\cdots(|x|+j-1)|}{(k+1)\cdots(j-1)} \le \frac{(|x|)_j}{j!} \le e^{|x|} j^{|x|}.$$

Combining the above with Lemma 7.9, we get the estimate

$$\max_{1\le j\le Mn+\ell} |b_{j,i,s}(x)| \le e^{|x|+|\alpha_i|} (Mn+\ell)^{1+s+|x|+|\alpha_i|} = e^{o(n)} \tag{60}$$

as n tends to infinity. We now estimate the coefficients $p_{n,j,\ell}$. For any integers k,j with $1 \le j \le Mn+\ell$ and $n \le k \le j+n$, we have

$$\binom{j+n}{k}\binom{k}{\ell} \le 2^{j+n} k^\ell \le 2^{(M+1)n+\ell}((M+1)n+\ell)^\ell = e^{o(n)} 2^{(M+1)n} \tag{61}$$

as n tends to infinity. Put $\alpha = \lceil \max_{1\le r\le d} |\alpha_r| \rceil$. Then

$$\left| \prod_{r=1}^{d} \binom{k-\alpha_r}{n}^{m_r+1} \right| \le \prod_{r=1}^{d} \binom{k+\alpha}{n}^{m_r+1} = \binom{k+\alpha}{n}^{M+1} \le \binom{(M+1)n+\ell+\alpha}{n}^{M+1}. \tag{62}$$

Finally, using Stirling's formula, we obtain, as n tends to infinity,

$$\begin{aligned}\binom{(M+1)n+\ell+\alpha}{n} = \frac{((M+1)n+\ell+\alpha)!}{n!(Mn+\ell+\alpha)!} &= e^{o(n)} \frac{((M+1)n+\ell+\alpha)^{(M+1)n+\ell+\alpha}}{n^n (Mn+\ell+\alpha)^{Mn+\ell+\alpha}} \\ &= e^{o(n)} \frac{((M+1)n)^{(M+1)n}}{n^n (Mn)^{Mn}} \\ &= e^{o(n)} \left(\frac{(M+1)^{M+1}}{M^M} \right)^n .\end{aligned}$$

Combining the above with (59)-(62), we deduce that

$$|p_{n,j,\ell}| \le e^{o(n)} \left(2\frac{(M+1)^{M+1}}{M^M} \right)^{(M+1)n}$$

as n tends to infinity, uniformly on $j \le Mn+\ell$. Together with (60), this yields $|Q_{n,i,s,\ell}(x)| \le e^{o(n)}\rho(M)^n$. □

9.2 p-adic absolute value of the Padé approximations

Let $n \ge 0$ be an integer. We keep the notation introduced at the beginning of Section 9. Recall that the functions μ and den are defined in (8). This section is devoted to estimating the p-adic absolute values of the Padé approximations $\mathfrak{R}_{n,i,s,\ell}(z) = P_{n,\ell}(z)R_{\alpha_i,s}(z) - Q_{n,i,s,\ell}(z)$ evaluated at some rational point x.

Proposition 9.2. *Let p be a prime number, $(i,s) \in \mathcal{S}$ and $x \in \mathbb{Q}$. Assume that*

$$|x|_p \cdot |\mu(\alpha_i)|_p > 1.$$

Then, for $\ell = 0, \dots, M$, the series $\mathfrak{R}_{n,i,s,\ell}(x)$ converges to an element of $\mathbb{Q}_p$, and

$$\limsup_{n\to\infty} \left|\mathfrak{R}_{n,i,s,\ell}(x)\right|_p^{1/n} \le p^{-1/(p-1)} \left| x\mu(\alpha_i)^{M+1} \prod_{k=2}^{d} \mu(\alpha_k)^{m_k+1} \right|_p^{-1}.$$

Remark 9.3. Since $\mu(\alpha_i)$ divides $\mu(\boldsymbol{\alpha})$ for each i, we have $|\mu(\alpha_i)|_p \ge |\mu(\boldsymbol{\alpha})|_p$. So, the statement of Theorem 9.2 still holds if we replace $\mu(\alpha_i)$ with $\mu(\boldsymbol{\alpha})$.

The proof of Proposition 9.2 uses the following notation. Given a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{Q}^m$ and $N \in \mathbb{N}$, we put

$$\mu_N(\boldsymbol{\alpha}) = \mathrm{den}(\boldsymbol{\alpha})^N \prod_{\substack{q:\text{prime} \\ q|\mathrm{den}(\boldsymbol{\alpha})}} q^{\lfloor N/(q-1) \rfloor}, \tag{63}$$

(recall that $\mu(\boldsymbol{\alpha})$ and $\mathrm{den}(\boldsymbol{\alpha})$ are defined in (8)). It follows easily from the definition that

$$|\mu_n(\boldsymbol{\alpha})|_p \ge |\mu(\boldsymbol{\alpha})|_p^n. \tag{64}$$

Note that for any integer ℓ and any rational number α, we have $\mathrm{den}(\alpha+\ell) = \mathrm{den}(\alpha)$. Consequently, we also have $\mu_n(\alpha+\ell) = \mu_n(\alpha)$. We will frequently use the following classical lemma, which is a direct consequence of [7, Lemma 2.2], to control the denominator of $(\alpha)_n/n!$.

Lemma 9.4. *Let n be a non-negative integer and $\alpha \in \mathbb{Q}$. Then, for $k = 0, \dots, n$, we have*

$$\mu_n(\alpha)\frac{(\alpha)_k}{k!} \in \mathbb{Z} \quad \text{and} \quad \mu_n(\alpha)\binom{\alpha}{k} \in \mathbb{Z}.$$

Lemma 9.5. *Let p be a prime number and $P(t) = \sum_{k=0}^{n} p_k \cdot (t)_k/k! \in \mathbb{Q}[t]$ be a rational polynomial of degree $n \geq 0$. Suppose that $p_0, \dots, p_n \in \mathbb{Z}$. Then, for any $(i,s) \in \mathcal{S}$, we have*

$$\left|\varphi_{\alpha_i,s}(P(t))\right|_p \leq (n+1)^{s-1} \left|\mu_{n+1}(\alpha_i)\right|_p^{-1}.$$

Proof. Fix $(i,s) \in \mathcal{S}$ and write $g_{i,s}(z) := \sum_{k=0}^{\infty} a_{i,s,k} z^k$ as in Definition 7.1. According to Lemma 7.2, we have

$$|\varphi_{\alpha_i,s}(P(t))|_p = \Big|\sum_{k=0}^{n} p_k(-1)^{k+1} a_{i,s,k}\Big|_p \leq \max_{0 \leq k \leq n} |a_{i,s,k}|_p. \tag{65}$$

We now estimate the p-adic norm of the coefficients $a_{i,s,k}$ thanks to the explicit formulas (32).

Case 1. Suppose $i = 1$. Then $s \geq 2$ and $\alpha_1 = 0$, and using (32) we obtain $a_{1,s,0} = \cdots = a_{1,s,s-2} = 0$ and the crude estimate

$$|a_{1,s,k}|_p \leq (k+1)^{s-1} \quad (k \geq s-1). \tag{66}$$

Note that we could easily get the better upper bound $((k+1)/(s-1))^{s-1}$ when $s \geq 2$ by using the inequality of arithmetic and geometric means, however it would not make a difference for our applications.

Case 2. Suppose $i \geq 2$. If $s = 1$, then using Lemma 9.4 together with (32), we obtain

$$|a_{i,1,k}|_p = \Big|\binom{\alpha_i}{k+1}\Big|_p \leq |\mu_{k+1}(\alpha_i)|_p^{-1}. \tag{67}$$

If $s \geq 2$, we combine again (32) with Lemma 9.4 and the estimates (66) to get

$$|a_{i,s,k}|_p = \Big|\sum_{j=0}^{k} \binom{\alpha_i}{j} a_{1,s,k-j}\Big|_p \leq |\mu_k(\alpha_i)|_p^{-1}(k+1)^{s-1}. \tag{68}$$

Finally, Eqs. (66)-(68) together with (65) yields $|\varphi_{\alpha_i,s}(P(t))|_p \leq (n+1)^{s-1}|\mu_{n+1}(\alpha_i)|_p^{-1}$. □

Proof of Proposition 9.2. Fix an integer ℓ with $0 \leq \ell \leq M$. Recall that by Theorem 7.3 (iii), we have

$$\mathfrak{R}_{n,i,s,\ell}(z) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} \frac{\varphi_{\alpha_i,s}\left((t+n)_{k-n} A_{n,\ell}(t)\right)}{z(z+1)\cdots(z+k)}.$$

Let k be an integer with $k \geq n$. Then

$$\left|\frac{k!}{(k-n)!}\frac{1}{x(x+1)\cdots(x+k)}\right|_p \leq |n!|_p |x|_p^{-k-1}. \tag{69}$$

It remains to estimate the p-adic norm of the coefficient $\varphi_{\alpha_i,s}\left((t+n)_{k-n}A_{n,\ell}(t)\right) \in \mathbb{Q}$. Note that the polynomial $P(t) = (t+n)_{k-n}A_{n,\ell}(t)$ has degree $nM + k + \ell$. Lemma 7.6 implies that

$$P(t) = \sum_{j=0}^{nM+k+\ell} (-1)^j p_j \sum_{h=0}^{j} \binom{j}{h} (-1)^{j-h} (n-h)_{k-n} A_{n,\ell}(-h) \frac{(z)_j}{j!}.$$

On the other hand, according to Lemma 9.4, we have for each integer $h \geq 0$

$$\Big(\prod_{r=2}^{d} \mu_n(\alpha_r)^{m_r+1}\Big) A_{n,\ell}(-h) = \Big(\prod_{r=2}^{d} \mu_n(\alpha_r)^{m_r+1}\Big)\binom{h}{\ell} \prod_{r=1}^{d} \binom{h-\alpha_r}{n}^{m_r+1} \in \mathbb{Z},$$

so that the polynomial $P(t)\prod_{j=2}^{d}\mu_n(\alpha_j)^{m_i+1}$ satisfy the hypothesis of Lemma 9.5. Using (64), we conclude that

$$|\varphi_{\alpha_i,s}(P(t))|_p \le (nM+k+\ell+1)^{s-1}|\mu(\alpha_i)|_p^{-(nM+k+\ell+1)}\Big|\prod_{j=2}^{d}\mu(\alpha_j)^{m_i+1}\Big|_p^{-n}.$$

Together with (69), it follows that the series $\mathfrak{R}_{n,i,s,\ell}(x)$ converges in $\mathbb{Q}_p$ as soon as $|x\mu(\alpha_i)|_p \ge p$. In that case, writing $Q(k) = (k(M+1)+\ell+1)^{s-1}$, we find

$$\begin{aligned}\big|\mathfrak{R}_{n,i,s,\ell}(x)\big|_p &\le |n!|_p\Big|\prod_{j=2}^{d}\mu(\alpha_j)^{m_i+1}\Big|_p^{-n}\sum_{k\ge n}|x|_p^{-k-1}Q(k)|\mu(\alpha_i)|_p^{-(nM+k+\ell+1)}\\ &= o(e^n)|n!|_p\Big|\prod_{j=2}^{d}\mu(\alpha_j)^{m_i+1}\Big|_p^{-n}|x|_p^{-n}|\mu(\alpha_i)|_p^{-n(M+1)}\end{aligned}$$

as n tends to infinity, where the implicit constant does not depend on n. To conclude, we raise both side of the above inequality to the power $1/n$ and use the well-known estimate

$$\lim_{n\to\infty}|n!|_p^{1/n} \le p^{-1/(p-1)}.$$

□

9.3 Denominators of the Padé approximants

Let $n \ge 0$ be an integer. We keep the notation introduced at the beginning of Section 9. We now estimate the denominators of $P_{n,\ell}(x)$ and $Q_{n,i,s,\ell}(x)$ for $x \in \mathbb{Q}^\times$. Given a positive integer N, we denote by d_N the least common multiple of $1,\dots,N$. Recall that the function μ (resp. μ_N) is defined in (8) (resp. (63)). We will prove the following result.

Proposition 9.6. *Let $x \in \mathbb{Q}\setminus\{0\}$. Put $m = \max_{1\le i\le d}\{m_i\}$ and define*

$$D_n(\boldsymbol{\alpha},x) = \mu_{Mn+M}(x)\Big(\prod_{i=2}^{d}\mu_n(\alpha_i)^{m_i+1}\Big)\cdot\mu_{Mn+M}(\boldsymbol{\alpha})\cdot d_{Mn+M}\Big(\prod_{k=1}^{m}d_{\lfloor (Mn+M)/k\rfloor}\Big).$$

Then, for any integer ℓ with $0\le\ell\le M$ and any $(i,s)\in\mathcal{S}$, we have

$$D_n(\boldsymbol{\alpha},x)P_{n,\ell}(x)\in\mathbb{Z} \quad \text{and} \quad D_n(\boldsymbol{\alpha},x)Q_{n,i,s,\ell}(x)\in\mathbb{Z}.$$

Furthermore,

$$\limsup_{n\to\infty}|D_n(\boldsymbol{\alpha},x)|^{1/n} = e^{\rho_\infty} \quad \text{and} \quad \limsup_{n\to\infty}|D_n(\boldsymbol{\alpha},x)|_p^{1/n} = e^{-\rho_p},$$

where

$$\begin{aligned}\rho_\infty &= M\Big(1+\sum_{j=1}^{m}\frac{1}{j}\Big)+\log(\eta),\\ \rho_p &= -\log|\eta|_p,\\ \eta &= \mu(x)^M\mu(\boldsymbol{\alpha})^M\prod_{j=2}^{d}\mu(\alpha_j)^{m_j+1}.\end{aligned}$$

The following lemma is a particular case of Shidlovsky's trick for estimating the denominators of coefficients of power of formal Laurent series (*confer* [20, lemma 7] and [1, p.17]). Recall that $(a_{i,s,k})_{k\ge0}$ are the coefficients of the series $g_{i,s}$ introduced in Definition 7.1.

Lemma 9.7. *Let $(i,s)\in\mathcal{S}$ and N be an integer with $N\geq(s-1)n$. Then, for $k=0,\dots,N$, we have*

$$\begin{aligned} a_{1,s,k}\prod_{j=1}^{s-1}d_{\lfloor N/j\rfloor}&\in\mathbb{Z}, && \text{if } i=1 \text{ and } s\geq 2,\\ a_{i,s,k}\cdot\mu_N(\alpha_i)\prod_{j=1}^{s-1}d_{\lfloor N/j\rfloor}&\in\mathbb{Z}, && \text{if } i\geq 2 \text{ and } s\geq 2,\\ a_{i,1,k}\cdot\mu_{N+1}(\alpha_i)&\in\mathbb{Z}, && \text{if } i\geq 2 \text{ and } s=1.\end{aligned}$$

Proof. Put $D_N=d_Nd_{\lfloor N/2\rfloor}\cdots d_{\lfloor N/(s-1)\rfloor}$. Suppose that $i=1$ and $s\geq 2$. In view of (32), it suffices to show that

$$\frac{D_N}{(\ell_1+1)\cdots(\ell_{s-1}+1)}\in\mathbb{Z} \tag{70}$$

for any $\ell_1,\dots,\ell_{s-1}$ with $\ell_1+\cdots+\ell_{s-1}\leq N-s+1$ and $\ell_1\geq\ell_2\geq\ldots\geq\ell_{s-1}\geq 0$. For such a choice of integers and for $j=1,\dots,s-1$, we have

$$j(\ell_j+1)\leq\ell_1+\cdots+\ell_{s-1}+s-1\leq N,$$

hence $\ell_j+1\leq\lfloor N/j\rfloor$. Thus, ℓ_j+1 is a factor of $d_{\lfloor N/j\rfloor}$. This implies (70).

If $i=1$ and $s\geq 2$, then we get the result by using the case $i=1$ together with (32) and Lemma 9.4. Similarly, we obtain the case $i\geq 2$ and $s=1$ by combining, again, (32) and Lemma 9.4. □

Proof of Proposition 9.6. Write $P_{n,\ell}(z)=\sum_{j=0}^{Mn+\ell}p_{j,\ell}\cdot(z)_j/j!$, where $p_{j,\ell}$ are defined in Theorem 7.3 (ii). Then, Lemma 9.4 yields

$$\prod_{i=2}^{d}\mu_n(\alpha_i)^{m_i+1}p_{j,\ell}\in\mathbb{Z}\quad\text{and}\quad\mu_{Mn+\ell}(x)\prod_{i=2}^{d}\mu_n(\alpha_i)^{m_i+1}P_{n,\ell}(x)\in\mathbb{Z}, \tag{71}$$

thus $D_n(\boldsymbol{\alpha},x)P_{n,\ell}(x)\in\mathbb{Z}$. We now prove the second statement. Fix $(i,s)\in\mathcal{S}$. Again, using Theorem 7.3 (ii), we have

$$Q_{n,i,s,\ell}(z)=\sum_{j=1}^{Mn+\ell}p_{j,\ell}\left(\sum_{k=0}^{j-1}\frac{(-1)^{k+1}a_{i,s,k}k!(z)_j}{j!(z)_{k+1}}\right) \tag{72}$$

where $a_{i,s,k}\in\mathbb{Q}$ are from Definition 7.1. By Lemma 9.7, for any integer k with $0\leq k<Mn+M$, we have

$$\mu_{Mn+M}(\alpha_i)\prod_{j=1}^{m}d_{\lfloor(Mn+M)/j\rfloor}a_{i,s,k}\in\mathbb{Z}. \tag{73}$$

Let k,j be two integers with $0\leq k<j$. Notice

$$\frac{k!(z)_j}{j!(z)_{k+1}}=\frac{k!}{j(j-1)\cdots(j-k)}\frac{(z+k+1)_{j-k-1}}{(j-k-1)!}=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1}{j-i}\frac{(z+k+1)_{j-k-1}}{(j-k-1)!},$$

(the last inequality arises from the partial fraction decomposition of $1/(x(x-1)\cdots(x-k))$ evaluated at $x=j$). Lemma 9.4 implies that

$$d_{Mn+M}\cdot\mu_{Mn+M}(x)\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1}{j-i}\frac{(x+k+1)_{j-k-1}}{(j-k-1)!}\in\mathbb{Z}. \tag{74}$$

We deduce from (71), (73), (74) combined with (72) that

$$\Big(\prod_{i=2}^{d}\mu_n(\alpha_i)^{m_i+1}\Big)\mu_{Mn+M}(\alpha_i)\Big(\prod_{j=1}^{m}d_{\lfloor (Mn+M)/j\rfloor}\Big)d_{Mn+M}\cdot\mu_{Mn+M}(x)Q_{n,i,s,\ell}(x)\in\mathbb{Z},$$

hence $D_n(\boldsymbol{\alpha},x)Q_{n,i,s,\ell}(x)\in\mathbb{Z}$.

Asymptotic estimate of $D_n(\boldsymbol{\alpha},x)$. First, note that

$$\lim_{N\to\infty}\mu_N(\boldsymbol{\alpha})^{1/N}=\mu(\boldsymbol{\alpha})\quad\text{and}\quad\lim_{N\to\infty}|\mu_N(\boldsymbol{\alpha})|_p^{1/N}=|\mu(\boldsymbol{\alpha})|_p.$$

The same goes by replacing $\boldsymbol{\alpha}$ with x. On the other hand, the prime number theorem (*confer* [41]) implies that $d_n=e^{n(1+o(1))}$, and $|d_n|_p\le 1$ for each n since d_n is an integer. We deduce from the definition of D_n and the above that

$$\lim_{n\to\infty}|D_n(\boldsymbol{\alpha},x)|^{1/n}=\mu(x)^M\left(\prod_{i=2}^{d}\mu(\alpha_i)^{m_i+1}\right)\cdot\mu(\boldsymbol{\alpha})^M\cdot e^M\left(\prod_{k=1}^{m}e^{M/k}\right)=e^{\rho_\infty},$$

$$\lim_{n\to\infty}|D_n(\boldsymbol{\alpha},x)|_p^{1/n}=|\mu(x)|_p^M\left(\prod_{i=2}^{d}|\mu(\alpha_i)|_p^{m_i+1}\right)\cdot|\mu(\boldsymbol{\alpha})|_p^M=e^{-\rho_p}.$$

□

10 Poincaré-Perron type recurrence

We keep the notation of Section 7. Recall that $d,m_1,\dots,m_d$ are positive integers, and

$$M=d-1+m_1+\cdots+m_d.$$

The goal of the section is to explain how we can improve the asymptotic estimates (58) of Proposition 9.1 for $\big(|Q_{n,i,s,\ell}(x)|^{1/n}\big)_{n\ge 0}$. As an application, we obtain the following improvement for $M\le 2$.

Proposition 10.1. *Let $x\in\mathbb{Q}$. Suppose that $M\le 2$. Then, for any $(i,s)\in\mathcal{S}$ and any integer ℓ with $0\le\ell\le M$, we have*

$$\limsup_{n\to\infty}|P_{n,\ell}(x)|^{1/n}\le 1,$$
$$\limsup_{n\to\infty}|Q_{n,i,s,\ell}(x)|^{1/n}\le 1.$$

Proposition 10.1 will be proven in Subsection 10.3. The idea behind the proof is to show that, for a fixed ℓ, the sequences $(P_{n,\ell}(x))_{n\ge0}$ and $(Q_{n,i,s,\ell}(x))_{n\ge0}$ satisfy a Poincaré-type recurrence of some order $J>0$

$$a_J(n)u(n+j)+a_{J-1}(n)u(n+j-1)+\cdots+a_0(n)u(n)=0 \tag{75}$$

for large enough n, where the coefficients $a_j(t)\in\mathbb{Q}[t]$ are polynomials and $a_J(t)\neq 0$. Then, we can apply Perron's Second Theorem below (see [34] and [37, Theorem C]) to estimate precisely the growth of a solution of the above recurrence. Computations for small values of M suggest that we can take $J=M+1$. Although the above approach works, each different integer ℓ with $0\le\ell\le M$ would lead to a different recurrence. In order to alleviate the computations, we will first reduce the problem to the study of some auxiliary sequences introduced in Subsection 10.1.

Theorem 10.2 (Perron's Second Theorem). *Let J be a positive integer. Assume that for $j = 0, \dots, J$ there exist a function $a_j : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ and $b_j \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} a_j(n) = b_j \in \mathbb{C},$$

with $b_J \neq 0$. Denote by $\lambda_1, \dots, \lambda_J$ the (not necessarily distinct) roots of the characteristic polynomial

$$\chi(z) = b_J z^J + b_{J-1} z^{J-1} + \cdots + b_0.$$

Then, there exist J linearly independent solutions $u_1, \dots, u_J$ of (75), such that, for each $j = 1, \dots, J$,

$$\limsup_{n\to\infty} |u_j(n)|^{1/n} = |\lambda_j|.$$

In particular, any solution u of (75) satisfies $\limsup_{n\to\infty} |u(n)|^{1/n} \leq \max_{1\leq j\leq J} |\lambda_j|$.

Remark 10.3. In the above theorem, there are no restriction on the roots of $\chi(z)$, whereas in Poincaré's Theorem and Perron's First Theorem, we ask that

$$|\lambda_i| \neq |\lambda_j| \text{ for } i \neq j, \tag{76}$$

see [37, Theorem A and B]. This is important to note as it seems that the characteristic polynomials we are dealing with never satisfy condition (76), see Figure 2.

10.1 Auxiliary sequences

Recall that $\Delta_{-1}(P(z)) = P(z-1) - P(z)$ for each $P(z) \in \mathbb{Q}[z]$. Similarly, $\boldsymbol{\Delta}_{-1}(P(t)) = P(t-1) - P(t)$. The $K[z]$-morphisms $\varphi_{\alpha_i,s}$ are defined in (31). For each integer $n \geq M$ and each $(i,s) \in \mathcal{S}$, put

$$\begin{aligned}
\widehat{A}_n(z) &= (-1)^{n-M} \prod_{r=1}^{d} \left(\frac{(z+\alpha_r)_n}{n!} \right)^{m_r+1}, \\
\widehat{P}_n(z) &= \Delta_{-1}^{n-M} \left(A_n(z) \right), \\
\widehat{Q}_{n,i,s}(z) &= \varphi_{\alpha_i,s} \left(\frac{\widehat{P}_n(z) - \widehat{P}_n(t)}{z-t} \right).
\end{aligned}$$

The goal of this section is to prove the following result.

Proposition 10.4. *Let $x \in \mathbb{Q}$. For each $(i,s) \in \mathcal{S}$ and each integer ℓ with $0 \leq \ell \leq M$, we have*

$$\begin{aligned}
\limsup_{n\to\infty} |P_{n,\ell}(x)|^{1/n} &\leq \max_{0\leq j\leq M} \limsup_{n\to\infty} |\widehat{P}_n(x-j)|^{1/n}, \\
\limsup_{n\to\infty} |Q_{n,i,s,\ell}(x)|^{1/n} &\leq \max_{0\leq j\leq M} \limsup_{n\to\infty} |\widehat{Q}_{n,i,s}(x-j)|^{1/n}.
\end{aligned}$$

We first establish some useful intermediate lemmas. The following is similar to Lemma 7.5.

Lemma 10.5. *Let j, n be non-negative integers with $j \leq M < n$. Then*

$$t^k \widehat{P}_n(t-j) \in \bigcap_{(i,s)\in\mathcal{S}} \ker \varphi_{\alpha_i,s} \qquad (0 \leq k < n-M). \tag{77}$$

Proof. Write $\widehat{A}_n(t-j) = \widehat{A}_{n-M}(t) Q_j(t)$ with $Q_j(t) \in \mathbb{Q}[t]$, and fix an integer k with $0 \leq k < n-M$. By Lemma 6.3 (iii), we have

$$t^k \widehat{P}_n(t-j) = t^k \boldsymbol{\Delta}_{-1}^{n-M} \big(\widehat{A}_{n-M}(t) Q_j(t) \big) \in \boldsymbol{\Delta}_{-1} \left(\mathbb{Q}[t] \prod_{r=1}^{d} (t+\alpha_r)^{m_r+1} \right) = \bigcap_{(i,s)\in\mathcal{S}} \ker \varphi_{\alpha_i,s},$$

the last equality coming from Lemma 8.7. Hence (77). □

Lemma 10.6. *Let $P(z), a(z) \in \mathbb{Q}[z]$ with $\deg a(z) = d \geq 0$, and set $\widetilde{P}(z) = a(z)P(z)$. Given $(i,s) \in \mathcal{S}$, we suppose that*

$$t^k P(t) \in \ker \varphi_{\alpha_i,s} \quad (k = 0, \dots, d-1).$$

Put

$$Q(z) = \varphi_{\alpha_i,s}\left(\frac{P(z)-P(t)}{z-t}\right) \quad \text{and} \quad \widetilde{Q}(z) = \varphi_{\alpha_i,s}\left(\frac{\widetilde{P}(z)-\widetilde{P}(t)}{z-t}\right).$$

Then, we have $\widetilde{Q}(z) = a(z)Q(z)$.

Proof. We follow the arguments in the proof of [25, Lemma 3.8]. First, note that the polynomial

$$b(t) = \frac{a(z)-a(t)}{z-t} \in \mathbb{Q}[z,t]$$

has degree at most $d-1$ in t. By hypothesis, the polynomial $b(t)P(t)$ belongs to the kernel of $\varphi_{\alpha_i,s}$. To conclude, it suffices to write

$$\frac{\widetilde{P}(z)-\widetilde{P}(t)}{z-t} = a(z)\frac{P(z)-P(t)}{z-t} + b(t)P(t),$$

and then to apply $\mathbb{Q}[z]$-linear morphism $\varphi_{\alpha_i,s}$ to the above identity. □

Proposition 10.7. *Let n, ℓ be integers with $0 \leq \ell \leq M$ and $2M \leq n$. For each $(i,s) \in \mathcal{S}$, we have*

$$P_{n,\ell}(z) = \sum_{j=0}^{M} a_j(n,z)\widehat{P}_n(z-j) \quad \text{and} \quad Q_{n,i,s,\ell}(z) = \sum_{j=0}^{M} a_j(n,z)\widehat{Q}_{n,i,s}(z-j),$$

where

$$a_j(n,z) = \sum_{k=0}^{M-j} \binom{n}{k}\binom{M-k}{j}\frac{(-1)^{j+k+\ell+Mn+M+n}}{\ell!}\Delta_{-1}^{k}\big((z-n+k)_\ell\big).$$

Proof. According to Lemma 6.1, and since $\Delta_{-1}^k((z)_\ell) = 0$ if $k > \ell$, we have

$$\begin{aligned}(-1)^{\ell+Mn+M+n}\ell! P_{n,\ell}(z) = \Delta_{-1}^{n}\big((z)_\ell \widehat{A}_n(z)\big) &= \sum_{k=0}^{n}\binom{n}{k}\Delta_{-1}^{k}\big((z-n+k)_\ell\big)\Delta_{-1}^{n-k}\big(\widehat{A}_n(z)\big)\\ &= \sum_{k=0}^{M}\binom{n}{k}\Delta_{-1}^{k}\big((z-n+k)_\ell\big)\Delta_{-1}^{M-k}\big(\widehat{P}_n(z)\big).\end{aligned}$$

Using the identity $\Delta_{-1}^m = \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}\tau_{-1}^{j}$, we find

$$\Delta_{-1}^{M-k}(\widehat{P}_n(z)) = \sum_{j=0}^{M-k}\binom{M-k}{j}(-1)^{M-k-j}\widehat{P}_n(z-j),$$

and rearranging the terms in the previous expression, we obtain the expected formula for $P_{n,\ell}(z)$. It follows that

$$Q_{n,i,s,\ell}(z) = \varphi_{\alpha_i,s}\left(\frac{P_{n,\ell}(z)-P_{n,\ell}(t)}{z-t}\right) = \sum_{j=0}^{M}\varphi_{\alpha_i,s}\left(\frac{a_j(n,z)\widehat{P}_n(z-j)-a_j(n,t)\widehat{P}_n(t-j)}{z-t}\right).$$

Note that for each j, the polynomial $a_j(n,z) \in \mathbb{Q}[z]$ has degree at most M (in the variable z), and that according to Lemma 10.5, we have $t^k\widehat{P}_n(t-j) \in \ker\varphi_{\alpha_i,s}$ for $k = 0,\ldots,M-1$. By Lemma 10.6, we conclude that

$$\varphi_{\alpha_i,s}\left(\frac{a_j(n,z)\widehat{P}_n(z-j) - a_j(n,t)\widehat{P}_n(t-j)}{z-t}\right) = a_j(n,z)\varphi_{\alpha_i,s}\left(\frac{\widehat{P}_n(z-j) - \widehat{P}_n(t-j)}{z-t}\right)$$
$$= a_j(n,z)\widehat{Q}_{n,i,s}(z-j).$$

□

Proof of Proposition 10.4. This is a consequence of Proposition 10.7, noticing that for a fixed $x \in \mathbb{Q}$, the polynomials $a_j(n,x)$ (in the variable n) satisfy

$$\max_{0\le j\le M} \limsup_{n\to\infty} |a_j(n,x)|^{1/n} = 1.$$

□

10.2 Poincaré-Perron recurrence

In view of Proposition 10.4, it remains to estimate the auxiliary sequences introduced in the previous section. This will be done by using Perron's Second Theorem (see Theorem 10.2). First, we reduce the problem by proving that it suffices to find a Poincaré-type recurrence for the sequence $\big(\widehat{P}_n(z)\big)_{n\ge M}$. The next result ensures that for each $(i,s) \in \mathcal{S}$, the sequence $\big(\widehat{Q}_{n,i,s}(z)\big)_{n\ge M}$ will also satisfy the same recurrence.

Proposition 10.8. *Suppose that there exist integers $J, d_0, \ldots, d_J \ge 0$ and sequences $\big(a_j(n,z)\big)_{\ge n}$ in $\mathbb{Q}[z]$ for $j = 0,\ldots,J$ with the following properties. For each integer $n \ge M$, we have $a_j(n,z) \in \mathbb{Q}[z]$ and*

$$\deg a_j(n,z) \le d_j \qquad (0 \le j \le J).$$

Assume that $\big(\widehat{P}_n(z)\big)_{n\ge M}$ satisfies the recurrence

$$\sum_{j=0}^{J} a_j(n,z)\widehat{P}_{n+j}(z) = 0 \tag{78}$$

for each $n \ge M$. Then, for any $(i,s) \in \mathcal{S}$, the sequence $\big(\widehat{Q}_{n,i,s}(z)\big)_{n\ge M}$ also satisfies the recurrence (78) *for each integer $n \ge M + \max\{0, N\}$, where $N = \max_{0\le j\le J}\{d_j - j\}$.*

Proof. By hypothesis, we have

$$0 = \sum_{j=0}^{J} \frac{\widetilde{P}_{j,n}(z) - \widetilde{P}_{j,n}(t)}{z-t}, \qquad \text{where } \widetilde{P}_{j,n}(z) = a_j(n,z)\widehat{P}_{n+j}(z).$$

Fix $(i,s) \in \mathcal{S}$. Using Lemmas 10.5 and 10.6 we find

$$0 = \varphi_{\alpha_i,s}\left(\sum_{j=0}^{J} \frac{\widetilde{P}_{j,n}(z) - \widetilde{P}_{j,n}(t)}{z-t}\right) = \sum_{j=0}^{J} \varphi_{\alpha_i,s}\left(\frac{\widetilde{P}_{j,n}(z) - \widetilde{P}_{j,n}(t)}{z-t}\right) = \sum_{j=0}^{J} a_j(n,z)\widehat{Q}_{n+j,i,s}(z),$$

for each $n \ge M$ such that $n + j - M \ge d_j$ for $j = 0,\ldots,J$, *i.e.* such that $n \ge M + N$. □

For small values of M, we can use Zeilberger's algorithm (see [36, Chapter 6]) to obtain non trivial recurrences of the form (78), see Section 10.3. Let us briefly explain how we could prove that such recurrences exist for any value of the parameter M. We will just give a sketch of the proof since we do not use this result in this paper (and since it would be a bit long to give all the details). Recall that for any integer $n \geq M$, we have

$$\widehat{P}_n(z) = \Delta_{-1}^{n-M}\left(A_n(z)\right) = \sum_{k\in\mathbb{Z}} F(z,n,k),$$

where $F(z,n,k)$ is the doubly hypergeometric term (with respect to n and k)

$$F(z,n,k) = \binom{n-M}{k}(-1)^k \prod_{r=1}^{d} \binom{z+\alpha_r-k-1}{n}^{m_r+1},$$

with the convention that $\binom{n-M}{k} = 0$ if either $k < 0$ or $k > n-M$. First, we could prove that there exist integers $I, J \geq 0$ and coefficients $a_{i,j}(z)$ independent of k, not all zero, which are polynomials in the parameters $z, n, \alpha_1, \ldots, \alpha_d$, such that

$$\sum_{i=0}^{I}\sum_{j=i}^{J} a_{i,j}(z)F(z,n-j,k-i) = 0 \qquad (\text{for } 0 \leq k \leq n-M \text{ and } n \geq M+J). \tag{79}$$

Essentially, this corresponds to [36, Theorem 4.4.1] and it seems that it suffices to follow Sister Celine's method. More explicitly, the steps of the algorithm are the following.

- Divide each term of (79) by $F(z,n,k) \neq 0$ and write $F(z,n-j,k-i)/F(z,n,k)$ as the ratio of some polynomial functions in the parameters $z, n, k, \alpha_1, \ldots, \alpha_d$.
- Multiply (79) by the least common multiple of the denominators of the above rational functions and write this expression as a polynomial in k. We can show that the degree of this polynomial is at most linear in I en J (more precisely it seems it is at most $(M+1)(I+J)$).
- Solve (in $\mathbb{Z}[z, n, \alpha_1, \ldots, \alpha_d]$) the system of linear equations in the unknown coefficients $a_{i,j}$ obtained by equating to zero the coefficients of each power of k. If $J \geq I$ we have at least $I(I+1)/2$ unknowns.

If I and J are large enough with $J = I$, then there are certainly more unknowns $a_{i,j}$ than equations, hence a non trivial solution satisfying (79). Here, our setting is slightly different from the setting in [36, Section 4 and 6] because we are dealing with the extra parameters $z, \alpha_1, \ldots, \alpha_d$, and we impose $a_{i,j}(z) = 0$ when $i > j$. Now, notice that $\sum_{k=0}^{n-M} F(z,n-j,k-i) = \widehat{P}_{n-j}(z)$ for any indices with $0 \leq i \leq j \leq J$. So, writing $b_j(z) = \sum_{i=0}^{I} a_{i,j}(z)$, and taking the sum from $k=0$ to $k = n-M$ in (79), we find

$$\sum_{j=0}^{J} b_j(z)\widehat{P}_{n-j}(z) = 0,$$

which is a recurrence of the form (78) (written differently). To conclude, we also must carefully justify that the coefficients $b_j(z)$ are not all equal to 0. Here, it seems that once again we can adapt the arguments used in the proof of [36, Theorem 6.2.1]. The idea is to consider a non trivial relation (79) with I minimal, and then to prove that if the polynomial $\sum_{j=0}^{J} b_j(z)Y^j$ is equal to 0, then we can find another relation of the form (79) with I replaced by $I-1$, which would contradict the minimality of I.

Numerical computations for small values of M indicate that the order J of the recurrence (78) is $M+1$.

10.3 Small values of M

Fix $x \in \mathbb{Q}$ and recall that $\alpha_1 = 0$. If $M \leq 2$, then $d = 1$ and $m_1 = M$. In this section we give the explicit Poincaré-Perron recurrence satisfied by $(\widehat{P}_n(z))_{n\geq M}$ for the above values of M. This will allow us to prove Proposition 10.1. The MAPLE's programs used for our computations are available at `https://apoels-math-u.net/Maple/polygamma_pade.zip`.

Case $d = 1$ and $m_1 = 1$. Then

$$\widehat{P}_n(z) = \sum_{k=0}^{n-1} F(z,n,k), \qquad \text{where } F(z,n,k) = (-1)^k \binom{n-1}{k}\binom{z+n-k-1}{n}^2.$$

Zeilberger's algorithm ensures that $(\widehat{P}_{n,0}(z))_{n\geq 1}$ satisfies the recurrence

$$(n^2+5n+6)u(n+2) - ((8+4z)n+6z+12)u(n+1) - (n^2+n)u(n) = 0. \tag{80}$$

Its characteristic polynomial is

$$\chi(T) = T^2 - 1$$

(independent of z), whose roots have modulus 1. By Proposition 10.8, the sequence $(\widehat{Q}_{n,1,2}(z))_{n\geq 1}$ also satisfies (80). Together with Theorem 10.2, this yields, for each $(i,s) \in \mathcal{S}$

$$\max_{0\leq j\leq 1} \limsup_{n\to\infty} |\widehat{P}_n(x-j)|^{1/n} \leq 1 \quad \text{and} \quad \max_{0\leq j\leq 1} \limsup_{n\to\infty} |\widehat{Q}_{n,1,2}(x-j)|^{1/n} \leq 1. \tag{81}$$

Case $d = 1$ and $m_1 = 2$. We have

$$\widehat{P}_n(z) = \sum_{k=0}^{n-1} F(z,n,k), \qquad \text{where } F(z,n,k) = (-1)^k \binom{n-1}{k}\binom{z+n-k-1}{n}^3.$$

Zeilberger's algorithm ensures that $(\widehat{P}_{n,0}(z))_{n\geq 1}$ satisfies the recurrence

$$a_3(n,z)u(n+3) + a_2(n,z)u(n+2) + a_1(n,z)u(n+1) + a_0(n,z)u(n) = 0, \tag{82}$$

where

$$\begin{aligned}
a_3(n,z) &= 2(3n+5)(n+4)(2n+7)(n+3)^2\\
a_2(n,z) &= -(3n+7)\big(9n^4+74n^3+3(9x^2+45x+127)n^2+(585x+117x^2+976)n+120(x^2+7+5x)\big),\\
a_1(n,z) &= 2n(n+1)(9n^3+48n^2+80n+43)\\
a_0(n,z) &= -n(3n+8)(n-1)(n+1)^2.
\end{aligned}$$

Its characteristic polynomial is

$$\chi(T) = 4T^3 - 9T^2 + 6T - 1 = (T-1)^2(4T-1),$$

(independent of z), whose largest roots have modulus 1. By Proposition 10.8, the sequence $(\widehat{Q}_{n,1,s}(z))_{n\geq 1}$ also satisfies (82) for $s = 2,3$. Together with Theorem 10.2, this yields, for each $(i,s) \in \mathcal{S}$

$$\max_{0\leq j\leq 2} \limsup_{n\to\infty} |\widehat{P}_n(x-j)|^{1/n} \leq 1 \quad \text{and} \quad \max_{0\leq j\leq 2} \limsup_{n\to\infty} |\widehat{Q}_{n,i,s}(x-j)|^{1/n} \leq 1. \tag{83}$$

Proof of Proposition 10.1. Suppose that $M \leq 2$ and fix $x \in \mathbb{Q}$. Then (81) and (83) combined with Proposition 10.4 yields the expected result. ☐

Case $M \geq 3$. It is possible to apply the above method when the parameters M is larger than 2. However, the computing time increases significantly at each step. Maple's computations suggest that the first characteristic polynomials are the following (we did the computation for $d = 1$):

M	characteristic polynomial $\chi(T)$
1	$(T-1)(T+1)$
2	$(T-1)^2(4T-1)$
3	$(T-1)(T+1)(27T^2+1)$
4	$(T-1)^2(-1+16T)(4T+1)^2$
5	$(T-1)(T+1)(3125T^4+625T^2+1)$
6	$(T-1)^4(-1+64T)(27T+1)^2$
7	$(T-1)(T+1)(823543T^6+6000099T^4+12005T^2+1)$

Figure 2: Expected characteristic polynomials for small values of M

It is seems that for $M=7$, the characteristic polynomial has two root of modulus $2.698\cdots>1$.

Remark 10.9. It would be desirable to determine the characteristic polynomial $\chi_M(T)$ as well as the modulus ρ_M of its largest roots for arbitrary M, since this would allow us to relax condition (86) of Theorem 11.1 below (which is a strong version of our main theorem) by replacing $g(M)$ defined in (84) with $\log\rho_M$.

11 Proof of our main theorem

In order to state a general version of our main Theorem 1.5, we need to introduce some notation. Let $d, m_1, \ldots, m_d$ be positive integers and $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_d)\in\mathbb{Q}^d$ with $\alpha_1=0$ and $\alpha_i-\alpha_j\notin\mathbb{Z}$ for any $i\neq j$. Put $m=\max_{1\le i\le d} m_i$, $M=\sum_{i=1}^d m_i+d-1$, $\boldsymbol{m}=(m_1,\ldots,m_d)$, and

$$\mathcal{S}=\{(i,s)\,;\,1\le i\le d \quad\text{and}\quad 1\le s\le m_i+1\}\setminus\{(1,1)\}.$$

Recall that $q_p=p$ if $p\ge 3$ and $q_p=4$ if $p=2$. The function μ is defined in (8). Set

$$f(\boldsymbol{\alpha},\boldsymbol{m})=g(M)+M\left(1+\sum_{j=1}^{m}\frac{1}{j}\right)+\log\left(\mu(\boldsymbol{\alpha})^M\prod_{i=2}^{d}\mu(\alpha_i)^{m_i+1}\right)-\log|\mu(\boldsymbol{\alpha})|_p,$$

where the function g is defined by $g(M)=0$ if $M=1,2$ and

$$g(M)=(M+1)\log\left(\frac{2(M+1)^{M+1}}{M^M}\right)\qquad\text{if } M>2. \tag{84}$$

Note that $(M+1)^M/M^M$ tends to e as M tends to infinity, so that

$$g(M)=M\log(M+1)+\mathcal{O}(M),$$

with an absolute implied constant.

Theorem 11.1. *Let p be a prime number and $x\in\mathbb{Q}$ satisfying*

$$|x|_p\ge q_p\max\{1,|\alpha_2|_p,\ldots,|\alpha_d|_p\} \tag{85}$$

and

$$\frac{\log p}{p-1}+\log|x|_p>M\log\left(\mu(x)|\mu(x)|_p\right)+f(\boldsymbol{\alpha},\boldsymbol{m}). \tag{86}$$

Then the $m_1+\cdots+m_d+1$ elements of $\mathbb{Q}_p$

$$1, G_p^{(2)}(x+\alpha_1),\ldots,G_p^{(m_1+1)}(x+\alpha_1),\ldots,G_p^{(2)}(x+\alpha_d),\ldots,G_p^{(m_d+1)}(x+\alpha_d)$$

are linearly independent of $\mathbb{Q}$.

Theorem 11.1 combined with (4) yields the following consequence, which is a refined version of Theorem 1.6.

THEOREM 11.2. *Let p and x satisfying the hypotheses of Theorem 11.1. Then* 1 *together with the* $m_1+\cdots+m_d$ *elements of* $\mathbb{Q}_p$

$$\omega(x+\alpha_i)^{1-s_i}\zeta_p(s_i,x+\alpha_i) \quad (1 \le i \le d \quad \textit{and} \quad 2 \le s_i \le m_i+1)$$

are linearly independent over $\mathbb{Q}$*, where* ω *denotes the Teichmüller character on* $\mathbb{Q}_p^\times$ *.*

The special case $d = m = 1$ was proved by Beukers, see [7, Theorem 9.2].

REMARK 11.3.

- Condition (86) is not optimal and could be relaxed by replacing g defined in (84) with a smaller function of M, see Remark 10.9.
- If the denominator of x is a power of p, then we have $\mu(x)|\mu(x)|_p = 1$, and Condition (86) becomes

$$\frac{\log p}{p-1} + \log|x|_p > f(\boldsymbol{\alpha},\boldsymbol{m}). \tag{87}$$

- In the case $d = 1$ and $m_1 = m \in \mathbb{Z}_{\ge 1}$, we have $M = m$ and $\mathrm{den}(\boldsymbol{\alpha}) = \mu(\boldsymbol{\alpha}) = 1$. Thus

$$f(\boldsymbol{\alpha},\boldsymbol{m}) = g(m) + m\left(1+\sum_{j=1}^{m}\frac{1}{j}\right) = 2m\log(m+1) + \mathcal{O}(m). \tag{88}$$

We will deduce Theorem 11.1 from the next theorem. For any non-negative integers ℓ, n with $0 \le \ell \le M$ and each $(i,s) \in \mathcal{S}$, the polynomials $P_{n,\ell}(z)$, $Q_{n,i,s,\ell}(z)$, and the Padé approximation $\mathfrak{R}_{n,i,s,\ell}(z) = P_{n,\ell}(z)R_{\alpha_i,s}(z) - Q_{n,i,s,\ell}(z)$ of $R_{\alpha_i,s}(z)$ are defined in Theorem 7.3. Recall that $R_{\alpha,s}$ is as in Definition 5.1.

THEOREM 11.4. *Let p be a prime number and $x \in \mathbb{Q}$ satisfying*

$$|x|_p \ge q_p \max_{1\le i\le d}\{1, |\alpha_i|_p\}. \tag{89}$$

Let $\beta, \rho_\infty, \rho_p, \delta$ be real numbers and $(D_n)_{n\ge 0}$ be a sequence of positive integers such that, for each $(i,s) \in \mathcal{S}$ and each $\ell = 0,\dots,M$, the numbers $D_nP_{n,\ell}(x)$ and $D_nQ_{n,i,s,\ell}(x)$ are integers (for each integer $n \ge 0$) and

$$\begin{aligned}
&\limsup_{n\to\infty} \max\left\{|P_{n,\ell}(x)|, |Q_{n,i,s,\ell}(x)|\right\}^{1/n} \le e^{\beta},\\
&\limsup_{n\to\infty} D_n^{1/n} \le e^{\rho_\infty},\\
&\limsup_{n\to\infty} |D_n|_p^{1/n} \le e^{-\rho_p},\\
&\limsup_{n\to\infty} |\mathfrak{R}_{n,i,s,\ell}(x)|_p^{1/n} \le e^{-\delta}.
\end{aligned}$$

Suppose that $\beta + \rho_\infty < \delta + \rho_p$. Then the $M+1$ elements $(1, R_{\alpha_i,s}(x))_{(i,s)\in\mathcal{S}}$ of $\mathbb{Q}_p$ are linearly independent over $\mathbb{Q}$.

PROOF. The proof is classical, see for example [14, Proposition 5.7] (in the case $K = \mathbb{Q}$, $v_0 = p$), which also provides an effective irrationality measure result. For the sake of completion we recall the arguments. First, note that the condition (89) ensures that $|x+\alpha_i|_p = |x|_p \ge q_p > 1$ for $i = 1,\dots,d$, so that $R_{\alpha_i,s}(x)$ converges p-adically for each $(i,s) \in \mathcal{S}$ by Lemma 5.2. If p does not divide the denominator $\mathrm{den}(\alpha_i)$ of α_i, we have $|x|_p| \cdot |\mu(\alpha_i)|_p = |x|_p > 1$. If p divides $\mathrm{den}(\alpha_i)$, then

$$|\mu(\alpha_i)|_p = |\mathrm{den}(\alpha_i)|_p p^{-1/(p-1)} = |\alpha_i|_p^{-1} p^{-1/(p-1)}.$$

and (89) implies that $|x|_p \cdot |\mu(\alpha_i)|_p \geq q_p p^{-1/(p-1)} > 1$. Consequently, the series $\mathfrak{R}_{n,i,s,\ell}(x)$ converges p-adically, see Proposition 9.2. By contradiction, suppose that $(1, R_{\alpha_i,s}(x))_{(i,s)\in\mathcal{S}}$ are linearly dependent over $\mathbb{Q}$. Then, there exists $(b, b_{i,s})_{(i,s)\in\mathcal{S}} \in \mathbb{Z}^{M+1} \setminus \{0\}$ such that

$$b + \sum_{(i,s)\in\mathcal{S}} b_{i,s} R_{\alpha_i,s}(x) = 0. \tag{90}$$

Given a positive integer n, define

$$\widehat{p}_{n,\ell} := D_n P_{n,\ell}(x) \quad \text{and} \quad \widehat{q}_{n,i,s,\ell} := D_n Q_{n,i,s,\ell}(x),$$

for each $(i,s) \in \mathcal{S}$ and each $\ell = 0, \ldots, M$. By hypothesis $\widehat{p}_{n,\ell}$ and $\widehat{q}_{n,i,s,\ell}$ are integers. Theorem 8.10 implies that the $(M+1) \times (M+1)$ matrix

$$\begin{pmatrix} \widehat{p}_{n,\ell} \\ \widehat{q}_{n,i,s,\ell} \end{pmatrix}_{\substack{0\leq\ell\leq M \\ (i,s)\in\mathcal{S}}}$$

is non-singular. Consequently, there exists an integer ℓ with $0 \leq \ell \leq M$ such that

$$K_n = K_n(\ell) := b\widehat{p}_{n,\ell} + \sum_{(i,s)\in\mathcal{S}} b_{i,s} \widehat{q}_{n,i,s,\ell}$$

is a non-zero integer. Our hypothesis implies that $\limsup_{n\to\infty} |K_n|^{1/n} \leq e^{\beta+\rho_\infty}$. Since $|K_n||K_n|_p \geq 1$, it follows that

$$\liminf_{n\to\infty} |K_n|_p^{1/n} \geq \liminf_{n\to\infty} |K_n|^{-1/n} \geq e^{-(\beta+\rho_\infty)}. \tag{91}$$

On the other hand, using (90), we find

$$\begin{aligned} K_n = K_n - \widehat{p}_{n,\ell}\Big(b + \sum_{(i,s)\in\mathcal{S}} b_{i,s} R_{\alpha_i,s}(x)\Big) &= \sum_{(i,s)\in\mathcal{S}} b_{i,s}(\widehat{q}_{n,i,s,\ell} - R_{\alpha_i,s}(x)\widehat{p}_{n,\ell}) \\ &= \sum_{(i,s)\in\mathcal{S}} b_{i,s} D_n \mathfrak{R}_{n,i,s,\ell}(x), \end{aligned}$$

from which we deduce the upper bound

$$\limsup_{n\to\infty} |K_n|_p^{1/n} \leq \limsup_{n\to\infty} |D_n \mathfrak{R}_{n,i,s,\ell}(x)|_p^{1/n} \leq e^{-\delta-\rho_p}.$$

Together with (91), we deduce that $e^{-(\beta+\rho_\infty)} \leq e^{-(\delta+\rho_p)}$, which contradicts our hypothesis $\beta + \rho_\infty < \delta + \rho_p$. □

Proof of Theorem 11.1. For each integer $n \geq 0$, set $D_n = D_n(\boldsymbol{\alpha}, x)$, where $D_n(\boldsymbol{\alpha}, x)$ is as in Proposition 9.6. Define $\beta = g(M)$ and

$$\begin{aligned} \rho_\infty &= M\Big(1 + \sum_{j=1}^{m} \frac{1}{j}\Big) + \log\Big(\mu(x)^M \mu(\boldsymbol{\alpha})^M \prod_{i=2}^{d} \mu(\alpha_i)^{m_i+1}\Big), \\ \rho_p &= -\log\Big|\mu(x)^M \mu(\boldsymbol{\alpha})^M \prod_{i=2}^{d} \mu(\alpha_i)^{m_i+1}\Big|_p, \\ \delta &= \log\left(p^{1/(p-1)} \cdot \Big|x\mu(\boldsymbol{\alpha})^{M+1} \prod_{i=2}^{d} \mu(\alpha_i)^{m_i+1}\Big|_p\right). \end{aligned}$$

Condition (86) is equivalent to $\beta + \rho_\infty < \delta + \rho_p$. By Propositions 9.1, 10.1, 9.6 and 9.2 (also see Remark 9.3) the hypotheses of Theorem 11.4 are satisfied with the above parameters. Therefore, the elements $(1, R_{\alpha_i,s}(x))_{(i,s)\in\mathcal{S}}$ of $\mathbb{Q}_p$ are linearly independent over $\mathbb{Q}$. To conclude, it suffices to notice that (12) and (14) combined with Lemma 5.2 yields, for each $(i,s)\in\mathcal{S}$ with $s\geq 2$,

$$R_{\alpha_i,s}(x) = R_s(x+\alpha_i) = -G_p^{(s)}(x+\alpha_i).$$

□

Proof of Theorem 1.5. Let d, m be positive integers, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d)\in\mathbb{Q}^d$ satisfying (5) and $x\in\mathbb{Q}$. Set $\boldsymbol{m} = (m,\dots,m)\in\mathbb{Z}^d$. Note that (5) implies that $\alpha_2,\dots,\alpha_d$ are not integers. Assume that there exist $a, a_2,\dots,a_d\in\mathbb{Z}$ coprime with p, and $r, r_2,\dots,r_d\in\mathbb{Z}_{\geq 1}$ such that $x = a/p^r$ and $\alpha_i = a_i/p^{r_i}$ for $i = 2,\dots,d$. If $d=1$, then by (87) and (88), condition (86) holds if

$$\log|x|_p \geq Cm\log(m+1),$$

for some large absolute constant $C\geq 1$. We conclude by using Theorem 11.1.
Suppose now that $d\geq 2$ and put $s = \max\{r_2,\dots,r_d\}$. With this notation, we have

$$|x|_p = p^r \quad \text{and} \quad \max\{1, |\alpha_2|_p,\dots,|\alpha_d|_p\} = p^s.$$

Furthermore, $\mathrm{den}(\boldsymbol{\alpha}) = p^s$ and $\mu(\boldsymbol{\alpha}) = p^{s+1/(p-1)}$, so that

$$\begin{aligned} f(\boldsymbol{\alpha},\boldsymbol{m}) &= g(M) + M\left(1+\sum_{j=1}^{m}\frac{1}{j}\right) + \log\left(\mu(\boldsymbol{\alpha})^M\prod_{i=2}^{d}\mu(\alpha_i)^{m+1}\right) - \log|\mu(\boldsymbol{\alpha})|_p \\ &= \mathcal{O}\big(M\log(M+1)\big) + \mathcal{O}\Big((M+(d-1)(m+1)+1)\log\mu(\boldsymbol{\alpha})\Big) \\ &= \mathcal{O}\Big(dm\big(\log(md+1)+s\log p\big)\Big), \end{aligned}$$

with absolute implicit constants. Therefore, there exists an absolute constant $C\geq 1$ such that condition (6) of Theorem 1.5 implies condition (86) of Theorem 11.1. Also note that in that case, inequality (85) is automatic if $C\geq 2$ (since $\log q_p\leq 2s\log p$). We conclude by using Theorem 11.1. □

12 Values of the p-adic Hurwitz zeta function

Let $g:\mathbb{Z}_{\geq 1}\to\mathbb{R}^+$ be the function defined at the beginning of Section 11. The following improves and generalizes Theorem 1.1.

THEOREM 12.1. *Let p be a prime number and m, r be positive integers. Assume that*

$$\left(r+\frac{1}{p-1}\right)\log p > g(m) + m + m\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right).$$

Then the $m+1$ elements of $\mathbb{Q}_p$:

$$1, \zeta_p(2,p^{-r}),\dots,\zeta_p(m+1,p^{-r})$$

are linearly independent over $\mathbb{Q}$.

PROOF. Denote by ω the Teichmüller character on $\mathbb{Q}_p^\times$. By Theorem 11.2 with $d=1$, $\boldsymbol{m}=m$, $\boldsymbol{\alpha}=0$ and $x = 1/p^r$, the $m+1$ numbers

$$1, \omega(p^{-r})^{-1}\zeta_p(2,p^{-r}),\dots,\omega(p^{-r})^{-m}\zeta_p(m+1,p^{-r})$$

are linearly independent over $\mathbb{Q}$. We conclude by noticing that $\omega(p^{-r}) = p^{-r}\in\mathbb{Q}$ by (15). □

Recall that the special case $d = m = 1$ was proved by Beukers in [7, Theorem 9.2]. Since $g(m) \sim m \log m$ as m tends to infinity, Theorem 12.1 implies Theorem 1.2. Similarly, Theorem 1.3 is an easy consequence of the following result.

THEOREM 12.2. *Let p be a prime number and a, b, m, δ be positive integers with $\delta = a - 3(m+1)b > 0$. Assume that*

$$\left(\delta - \frac{3m+2}{p-1}\right) \log p > g(2m+1) + (2m+1)\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right). \tag{92}$$

Then the $2m+1$ elements of $\mathbb{Q}_p$:

$$1, \zeta_p(2, p^{-a}), \ldots, \zeta_p(m+1, p^{-a}), \zeta_p\big(2, p^{-a} + p^{-b}\big), \ldots, \zeta_p\big(m+1, p^{-a} + p^{-b}\big)$$

are linearly independent over $\mathbb{Q}$.

The hypothesis $\delta > 0$ ensures that (92) is satisfied for large enough p (with m, a, b fixed).

PROOF. Set $x = p^{-a}$, $\boldsymbol{m} = (m, m)$, $\alpha_2 = p^{-b}$ and $\boldsymbol{\alpha} = (0, \alpha_2)$. With the notation of Theorem 11.1, we have

$$\mu(\boldsymbol{\alpha}) = \mu(\alpha_2) = p^{b+1/(p-1)}.$$

A short computation also yields $M = 2m+1$ (since $d = 2$) and

$$\begin{aligned} f(\boldsymbol{\alpha}, \boldsymbol{m}) &= g(M) + M\left(1 + \sum_{j=1}^{m} \frac{1}{j}\right) + \log\left(\mu(\boldsymbol{\alpha})^M \prod_{i=2}^{d} \mu(\alpha_i)^{m+1}\right) - \log|\mu(\boldsymbol{\alpha})|_p \\ &= g(M) + (2m+1)\left(1 + \sum_{j=1}^{m} \frac{1}{j}\right) + \log\left(\mu(\boldsymbol{\alpha})^{3m+3}\right). \end{aligned}$$

Consequently, condition (86) of Theorem 11.1 is equivalent to condition (92). Also note that if $\delta > 0$, then $a \geq 2 + b$, so that $|x|_p \geq p^2 |\alpha_2|_p \geq q_p |\alpha_2|_p$, as required in Theorem 11.1. By Theorem 11.2, we conclude that

$$1, \lambda^{-1}\zeta_p(2, p^{-a}), \ldots, \lambda^{-m}\zeta_p(m+1, p^{-a}), \mu^{-1}\zeta_p\big(2, p^{-a} + p^{-b}\big), \ldots, \mu^{-m}\zeta_p\big(m+1, p^{-a} + p^{-b}\big)$$

are linearly independent over $\mathbb{Q}$, with $\lambda = \omega(p^{-a})$ and $\mu = \omega(p^{-a} + p^{-b})$, and where ω denotes the Teichmüller character on $\mathbb{Q}_p^{\times}$. Finally, $\lambda, \mu = \pm 1$ if $p = 2$, and (15) implies that $\lambda = \mu = p^{-a} \in \mathbb{Q}$ if $a > b$ and $p \geq 3$, hence the expected result. □

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