The complex case of Schmidt's going-down Theorem

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Abstract

In 1967, Schmidt wrote a seminal paper [10] on heights of subspaces of $\mathbb{R}^n$ or $\mathbb{C}^n$ defined over a number field K, and diophantine approximation problems. The going-down Theorem – one of the main theorems he proved in his paper – remains valid in two cases depending on whether the embedding of K in the complex field $\mathbb{C}$ is a real or a complex non-real embedding. For the latter, and more generally as soon as K is not totally real, at some point of the proof, the arguments in [10] do not exactly work as announced. In this note, Schmidt's ideas are worked out in details and his proof of the complex case is presented, solving the aforementioned problem. Some definitions of Schmidt are reformulated in terms of multilinear algebra and wedge product, following the approaches of Laurent [5], Bugeaud and Laurent [1] and Roy [7], [8].
In [5] Laurent introduces in the case $K = \mathbb{Q}$ a family of exponents and he gives a series of inequalities relating them. In Section 5 these exponents are defined for an arbitrary number field K. Using the going-up and the going-down Theorems Laurent's inequalities are generalized to this setting.

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1 Introduction

In a paper [10] written in 1967, Schmidt generalizes the basic diophantine approximation problem « given a real number α, how « well » can it be approximated by rational numbers ? » as follows. Let A be a subspace of a Euclidean or unitary space G^n of dimension n. Suppose that A has dimension $0 < d < n$. How « well » can A be approximated by subspaces B of dimension e defined over a given number field K ? Formulating precisely what « well » means requires some work. Schmidt binds two different notions that are recalled in Section 2 below : A is « well » approximated by B if on the one hand A and B are « close » (Schmidt uses several angles of inclination to measure this « closeness », cf. Proposition 2 and following definitions), and on the other hand B is not too « complicated » (Schmidt uses the notion of the *height* of a subspace to measure its « complicatedness », cf. (2.1) and (2.4)).
In his article, Schmidt establishes several transference theorems of the Perron-Khintchine-type (see for example [3], [4], [6]). These theorems lead to the conclusion that if a subspace A can be well approximated by subspaces of dimension e $(0 < e < n)$, then it can also be well approximated by subspaces of any given dimension e'. Schmidt's

going-down Theorem ([10] Theorem 10) is one of these transference theorems (treating the case $e' < e$) and is useful to prove diophantine approximation theorems (as [10] Theorem 13 for example). More recently this work was revisited by Laurent [5] and Bugeaud and Laurent [1] in the case where A is a one-dimensional subspace of $\mathbb{R}^n$ and $K = \mathbb{Q}$. Laurent introduces a family of approximation exponents to points in $\mathbb{R}^n$ by linear subspaces and using going-up and going-down Theorems he proves a series of inequalities relating these exponents [5]. Roy shows [8] that the going-up and going-down transference inequalities of Schmidt and Laurent describe the full spectrum of these exponents. In Section 5 these exponents are generalized for an arbitrary number field K. Using the going-up and the going-down Theorems one shows that Laurent's inequalities remain valid for the aforementioned generalized exponents.

The going-down Theorem remains valid in both the real case (a) and the complex (non-real) case (b) (see below for details). At some point of the proof for (b), Schmidt's arguments do not exactly work as announced; as the referee pointed out, this happens also in case (a) if K is not totally real (see Remarks 4 and 5 for more technical details). The main goal of this note is to work out Schmidt's ideas in details and solve this problem.
In case (a), G^n denotes Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, K a number field embedded in $\mathbb{R}$, and $q = 1$.
In case (b), G^n denotes unitary space $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$, K a number field embedded in $\mathbb{C}$ (but this embedding is not real), and $q = 2$.
In this paper $C_1, C_2, \dots$ will be positive constants depending only on K and n but independent from the subspaces $A^d, B^e, \dots$ considered. We also keep as far as possible the same numbering as Schmidt (so that the numbering does not start with C_1). The notation A^d (or $B^d, S^d, \dots$) will always mean that A^d (or $B^d, S^d, \dots$) has dimension d.

Theorem 1 (Schmidt's going-down Theorem)**.**
Let A^d, B^e be subspaces of G^n with B^e defined over K and of height $H(B^e) \leq H$ (where $H \geq 1$ is a fixed constant). Let $1 \leq h \leq f' = \min(d, e-1)$, $c \geq 1$ and assume that

$$H(B^e)\omega_i^q(A^d, B^e) \leq c^q H^{-(qy_i - 1)} \quad (i = 1, \dots, h). \tag{1.1}$$

where $y_1 \geq \dots \geq y_h \geq (qh)^{-1}$. Put $y = y_1 + \dots + y_h$ and assume

$$y_i' := y_i e(qy + e - 1)^{-1} \geq q^{-1} \quad (i = 1, \dots, h). \tag{1.2}$$

Then there is a subspace $B^{e-1} \subset B^e$, defined over K, of height

$$H(B^{e-1}) \leq C_5 H(B^e) H^{(qy-1)/e} \leq C_5 H^{(e+qy-1)/e} =: H'$$

having

$$H(B^{e-1})\omega_i^q(A^d, B^{e-1}) \leq C_6 c^q H^{-(qy_i'-1)(qy+e-1)/e} = C_7 c^q H'^{-(qy_i'-1)} \quad (i = 1, \dots, h),$$

whence

$$\omega_i(A^d, B^{e-1}) \leq C_8 c H(B^{e-1})^{-y_i'} \quad (i = 1, \dots, h).$$

On the other hand, if instead of (1.1)*,*

$$\omega_i(A^d, B^e) = 0 \quad (i = 1, \dots, h), \tag{1.3}$$

put

$$y_0' := e(qh)^{-1}.$$

Then, for any given $H' \geq C_9 H$ there is a subspace $B^{e-1} \subset B^e$, defined over K, of height $H(B^{e-1}) \leq H'$, having

$$H(B^{e-1})\omega_i^q(A^d, B^{e-1}) \leq C_{10} H^{qy_0'} H'^{-(qy_0'-1)} \quad (i = 1, \dots, h),$$

whence

$$\omega_i(A^d, B^{e-1}) \leq C_{11} H^{y_0'} H(B^{e-1})^{-y_0'} \quad (i = 1, \dots, h).$$

In this theorem, $C_5, \dots, C_{11}$ are positive constants which depend on $K, n, y_1, \dots, y_h$ but not on A^d, B^e, B^{e-1}, H, c.

In Section 2 we recall the definitions of the *height* of a subspace and of the functions ω_i which are used to measure the closeness of two subspaces. We also recall results of [10] which we will need in the proof of Theorem 1. Some definitions of Schmidt are reformulated in terms of multilinear algebra and wedge product – this is the case of the definition of the height of a subspace for instance – following the approaches of Laurent [5], Bugeaud and Laurent [1] and Roy [7], [8]. In Section 3 we introduce specific notation for the complex case in order to avoid confusion between the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum x_i \overline{y_i}$ and the bilinear form $\varphi(\mathbf{x}, \mathbf{y}) = \sum x_i y_i$ (which are denoted in the same way in [10]). In Section 4 we present Schmidt's proof of the going-down Theorem in the complex case, solving in Section 4.2 the problem alluded to above (see also Remarks 4 and 5 in Section 4.4). In Section 5 the generalization of Laurent's exponents is given.

2 Multilinear algebra, distance and height of subspaces

In this section one reformulates some definitions of Schmidt [10] – among others the height $H(S)$ of a subspace S and the quantities $\omega_i(A, B)$ which characterize the distance between two subspaces A and B – in terms of multilinear algebra and wedge product. This approach has already been investigated by Laurent in [5] and Bugeaud and Laurent [1] in order to give another proof of the going-up and going-down transfers in the case $K = \mathbb{Q}$ with A^d of dimension $d = 1$. See also Roy [7] and [8] for further examples of the use of such tools in the context of parametric geometry of numbers.

Let $\mathbb{L} = \mathbb{R}$ or $\mathbb{C}$ and let $n \in \mathbb{N}^*$. We endow $G^n := \mathbb{L}^n$ with its usual structure of inner product space. Let $\langle \cdot, \cdot \rangle$ denotes the canonical inner product on G^n (if $\mathbb{L} = \mathbb{C}$, we ask for the linearity of the first argument) and $|| \cdot ||$ its associated norm. If we fix an integer m, $1 \leq m \leq n$, we always endow the vector space $\bigwedge^m(G^n)$ with the unique structure of inner product space such that, for any orthonormal basis $(e_1, \dots, e_n)$ of G^n, the products $e_{i_1} \wedge \cdots \wedge e_{i_m}$ $(i_1 < \cdots < i_m)$ form an orthonormal basis of $\bigwedge^m(G^n)$. We still denote by $\langle \cdot, \cdot \rangle$ its inner product and by $|| \cdot ||$ the associated norm. (Note that with this notation, we have $D(X_1, \dots, X_m) = ||X_1 \wedge \cdots \wedge X_m||$ for any $X_1, \dots, X_m \in G^n$, where $D(X_1, \dots, X_m) = \Big(\det\Big(\langle X_i, X_j \rangle\Big)_{i,j}\Big)^{1/2}$ denotes the *generalized determinant* of $(X_1, \dots, X_m)$, see [10] for more details about this notion).

Let K be an algebraic number field of degree $[K : \mathbb{Q}] = p$ and $\mathcal{O}_K$ be the ring of integers of K. Let $\sigma_1, \dots, \sigma_p$ be the different embeddings of K into the field $\mathbb{C}$ of complex numbers. For $\xi \in K$, put $\xi^{(i)}$ for the image of ξ under σ_i. Similarly, if $X = (\xi_1, \dots, \xi_n) \in K^n$, put $X^{(i)} = (\xi_1^{(i)}, \dots, \xi_n^{(i)})$. Let S^d a subspace of K^n of dimension d. Let $(X_1, \dots, X_d)$ be a basis of S^d and form the matrix M with row vectors

$X_1, \dots, X_d$. Let $\mathfrak{a}$ be the fractional ideal of K generated by the $\binom{n}{d}$ determinants of all $d \times d$– submatrices of M. The *height* of S^d is defined by

$$H(S^d) = N(\mathfrak{a})^{-1} \prod_{j=1}^{p} ||X_1^{(j)} \wedge \cdots \wedge X_d^{(j)}||, \tag{2.1}$$

where $N(\mathfrak{a}) = N_{K/\mathbb{Q}}(\mathfrak{a}) \in \mathbb{Q}^+$ denotes the *norm* of the ideal $\mathfrak{a}$. This definition does not depend of the choice of the basis $(X_1, \dots, X_d)$. See [10] §1 for more explanations about this notion.

We suppose now that K is embedded in $\mathbb{L}$. We denote by $\bigwedge^m(\mathcal{O}_K^n)$ $(1 \leq m \leq n)$ the free $\mathcal{O}_K$-module of rank $\binom{n}{m}$ spanned by the products $x_1 \wedge \cdots \wedge x_m$ with $x_1, \dots, x_m \in \mathcal{O}_K^n$. A subspace S of $\mathbb{C}^n$ is said to be defined over K if it is defined by linear equations with coefficients in K (or equivalently, if there is a basis of S with coordinates in K). If S is defined over K, one can considerer its height as the height of $S \cap K^n$. If $(X_1, \dots, X_d) \in K^n$ form a basis of S^d, the associated fractional ideal $\mathfrak{a}$ defined above is the fractional ideal generated by the coordinates of $X_1 \wedge \cdots \wedge X_d$ with respect to a basis of $\bigwedge^m(\mathcal{O}_K^n)$. Note that if $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, we may suppose that $(X_1, \dots, X_d)$ form a basis of $S^d \cap \mathbb{Z}^n$ (and so, that it can be extended to a basis $(X_1, \dots, X_n)$ of $\mathbb{Z}^n$, which is equivalent to asking that $X_1 \wedge \dots X_d$ is a primitive vector, *i.e* $N(\mathfrak{a}) = 1$). In the general case since $\mathcal{O}_K$ is not necessarily a principal ring, $S^d \cap \mathcal{O}_K^n$ may not be a free $\mathcal{O}_K$-module. However, the ideal class group of K is finite and using a system of representatives consisting of integral ideals, it can be proved that $X_1, \dots, X_d$ may be chosen such that $X_1, \dots, X_d \in \mathcal{O}_K^n$ and $N(\mathfrak{a}) \leq C$ where $C > 0$ depends of K only.
Formula (2.4) in § 2 allows one to consider the height of some subspace more geometrically.

Finally, one has to introduce the functions ω_i used by Schmidt to measure the "closeness" of two subspaces A and B of a Euclidean or unitary space. We define the (projective) distance between two non-zero vectors X and Y of G^n by

$$\text{dist}\,(X, Y) := \frac{||X \wedge Y||}{||X||\,||Y||}.$$

Note that in [10] $\text{dist}\,(X, Y)$ is denoted by $\omega(X, Y)$. It satisfies the triangle inequality

$$\text{dist}\,(X, Z) \leq \text{dist}\,(X, Y) + \text{dist}\,(Y, Z) \quad X, Y, Z \in G^n \setminus \{0\}.$$

For $X \in G^n \setminus \{0\}$ and a subspace $B^e \neq \{0\}$ of G^n, we define the distance from X to B^e by

$$\text{dist}\,(X, B^e) = \inf_{Y \in B^e \setminus \{0\}} \text{dist}\,(X, Y)\,.$$

Note that

$$\text{dist}\,(X, B)^2 + \text{dist}\left(X, B^{\perp}\right)^2 = 1 \tag{2.2}$$

for every subspace B of dimension $0 < e < n$ (see [10, Section 8] formula (8)). Note once again that in [10] $\text{dist}\,(X, B^e)$ is denoted by $\omega(X, B^e)$ and that this infimum is in fact a minimum.

Definition 1. Let A^d and B^e be subspaces of G^n of dimensions d and e respectively, with $f := \min(d, e) > 0$. Set

$$\omega_i\left(A^d, B^e\right) := \inf_{F^i \subset A^d} \sup_{X \in F^i \setminus \{0\}} \text{dist}\,(X, B^e)\,,$$

for $i = 1, \dots, f$. Here F^i refers to an arbitrary subspace of A of dimension i.

Intuitively, the smaller the ω_i are, the closer A^d and B^e are. These quantities are the same as those introduced by Schmidt in [10, §8] (it is a direct consequence of Schmidt's definitions, his Lemma 12 and Lagrange's identity $||X||^2||Y||^2 = |\langle X,Y\rangle|^2 + ||X\wedge Y||^2$). In particular, we have the useful following result (see [10, Theorem 4 on page 443] noting that $\lambda_i = \sqrt{1-\omega_i^2}$) :

Proposition 2. *Let A^d and B^e be subspaces of G^n of dimensions d and e respectively, with $f := \min(d,e) > 0$. Then there are orthonormal bases $X_1,\ldots,X_d$ and $Y_1,\ldots,Y_e$ of A^d, B^e respectively, and reals $0 \le \omega_1 \le \cdots \le \omega_f \le 1$ such that*

$$\operatorname{dist}(X_i,Y_j) = \begin{cases} \omega_i & \text{if } i=j \\ 1 & \text{otherwise} \end{cases} \qquad (1\le i\le d,\ 1\le j\le e).$$

The numbers $\omega_1,\ldots,\omega_f$ are independent of any freedom of choice in X_i, Y_j and are invariant under unitary transformations applied simultaneously to A^d, B^e. Moreover, one has

$$\omega_i = \omega_i\left(A^d,B^e\right) \quad (1\le i\le f).$$

If $d+e\le n$, set

$$\mu(A^d,B^e) := \prod_{k=1}^{f}\omega_k(A^d,B^e).$$

(Although we will not use it in this paper, if $d+e>n$, $\mu(A^d,B^e)$ can be defined as $\prod_{k=1}^{f}\omega_{k+g}(A^d,B^e)$ where $g := d+e-n$. See Sections 7 and 8 of [10] for more details).

The next and last proposition is an equivalent definition of μ in the case $d+e\le n$ (see [10, § 6-8], especially formula (7)).

Proposition 3. *If $(X_1,\ldots,X_d)$ and $(Y_1,\ldots,Y_e)$ are arbitrary bases of A^d, B^e, respectively, and $d+e\le n$, one has*

$$\mu(A^d,B^e) = \frac{||X_1\wedge\cdots\wedge X_d\wedge Y_1\wedge\cdots\wedge Y_e||}{||X_1\wedge\cdots\wedge X_d||\ ||Y_1\wedge\cdots\wedge Y_e||}.$$

This formula generalizes the formula (4.1) of [1], which describes the special case $d=f=1$ (in this case, $\mu(A^d,B^e) = \operatorname{dist}(X_1,B^e)$).

For $n\in\mathbb{N}^*$, E^n denotes the Euclidean space $\mathbb{R}^n$ with its canonical scalar product. A *lattice* of E^n will mean a discrete group of vectors of E^n (not necessarily cocompact). The *rank* of a lattice is the maximal number of linearly independent vectors of the lattice. Define the determinant of a lattice Λ of rank m by $d(\Lambda) = ||X_1\wedge\cdots\wedge X_m||$ where $X_1,\ldots,X_m$ are basis vectors of Λ if $m>0$, and by $d(\Lambda)=1$ if $\Lambda=\{0\}$.
Suppose now that $K\subset\mathbb{C}$ but $K\not\subset\mathbb{R}$. Let $p = r_1+2r_2$ and $\xi^{(2r_2+j)}$ be real for $1\le j\le r_1$, $\xi^{(j+1)}$ the complex conjugate of $\xi^{(j)}$ for $1\le j\le 2r_2-1$, j odd, and every $\xi\in K$. We may assume σ_1 to be the identity map, σ_2 the complex conjugate map (such that $\sigma_1(\xi),\sigma_2(\xi)$ for $\xi\in\mathbb{C}$ – not necessarily in K – can also be considered). Put

$$\Delta = 2^{-r_2}|\delta|^{1/2}$$

where δ is the *discriminant* of K. Set

$$\xi^{[i]} = \begin{cases} \operatorname{Re}\xi^{(i)} & \text{if } 1\le i\le 2r_2 \text{ and } i \text{ odd,} \\ \operatorname{Im}\xi^{(i)} & \text{if } 1\le i\le 2r_2 \text{ and } i \text{ even,} \\ \xi^{(i)} & \text{if } 2r_2+1\le i\le p. \end{cases}$$

Here, Re and Im denote real and imaginary parts. Given $1\le i\le p$ and $X = (\xi_1,\ldots,\xi_n)\in K^n$, write $X^{(i)} := (\xi_1^{(i)},\ldots,\xi_n^{(i)})$ and $X^{[i]} := (\xi_1^{[i]},\ldots,\xi_n^{[i]})$. For $i=1,2$

$X^{(i)}$ and $X^{[i]}$ are defined for any $X \in \mathbb{C}^n$. Then, notice that for all $X \in \mathbb{C}^n$ one has $X = X^{[1]} - iX^{[2]}$. Let $\rho : K^n \to E^{np}$ be the $\mathbb{Q}$-linear map defined by

$$\rho(X) = (X^{[1]}, \dots, X^{[p]}) \in E^{np}. \tag{2.3}$$

It is the same map ρ as the one defined by Schmidt [10, p. 435] if one rearranges its coordinates; this does not change the main property (2.4) recalled below.
Let S^d be a subspace of K^n of dimension d, and $\mathcal{O}_K(S^d)$ be the subset of all of $X \in S^d$ whose components are in $\mathcal{O}_K$. Then $\Lambda(S^d) := \rho(\mathcal{O}_K(S^d))$ is a lattice in E^{np} of rank dp (see [10, §3]). Moreover Theorem 1 on page 435 of [10] asserts that

$$H(S^d) = \Delta^{-d} d\big(\Lambda(S^d)\big), \tag{2.4}$$

where $d\big(\Lambda\big)$ denotes the determinant of the lattice Λ.

3 Specific notation in the complex case

In the setting of Section 2 let us assume now that K is non-real and $G = \mathbb{C}^n$. We shall distinguish carefully the sesquilinear scalar product $\langle X, Y\rangle$ on $\mathbb{C}^n$ and the canonical bilinear form $\varphi(X, Y)$ on $\mathbb{C}^n$ or K^n. By contrast, both are denoted by XY in [10]. Of course in the real case $\langle X, Y\rangle$ and $\varphi(X, Y)$ coincide. Let $\langle\cdot,\cdot\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the inner product defined by

$$\langle (x_i)_i, (y_i)_i\rangle = \sum_{i=1}^n x_i \overline{y_i}.$$

If W is a subspace of $\mathbb{C}^n$, $W^\perp$ denotes the orthogonal complement of W for $\langle\cdot,\cdot\rangle$. Let K' be the complex conjugate field of K, $K' = \{\overline{z}\ ;\ z \in K\}$. If W is defined over K, note that $W^\perp$ is defined over K', and generally not over K. If S is a subspace of $\mathbb{C}^n$ defined over K, it follows easily from the definition of the height that $H(S') = H(S)$, where S' denotes the set of all $\overline{z}$ with $z \in S$, which is a subspace defined over K'. If $\varphi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denotes the bilinear form defined by

$$\varphi((x_i)_i, (y_i)_i) = \sum_{k=1}^n x_i y_i,$$

and $W^{\varphi,\perp}$ denotes its orthogonal complement with respect to φ, one can show that

$$H(S^{\varphi,\perp}) = H(S)$$

for all subspaces S of $\mathbb{C}^n$ defined over K (see [10] Eq. (4) on page 433 and [2] Theorem 1 on page 294, although it is not expressed in the same language). We may in particular deduce from the last statement this useful result :
Let S be a subspace of $\mathbb{C}^n$ defined over K. The subspace $S^\perp$ is defined over K' and satisfies

$$H(S) = H(S^\perp). \tag{3.1}$$

4 Proof of the going-down Theorem in the complex case

Proof. In this section one proves Schmidt's going-down Theorem (that is, Theorem 1 in the introduction), in case (b). In other words, K is a number field embedded in $\mathbb{C}$, with $K \not\subset \mathbb{R}$. We can suppose that $G^n = \big(\mathbb{C}^n, \langle\cdot,\cdot\rangle\big)$; we keep the notation of Sections

2 and 3. Let K' be the complex conjugate field of K, $K' = \{\overline{z}\ ;\ z \in K\}$.
Write $p = r_1 + 2r_2$. Notation $\xi^{(i)}$, $\xi^{[i]}$, ρ (cf. (2.3))... will be used with respect to K' ($\sigma_1, \ldots, \sigma_p$ denote the different isomorphisms of K' into the field $\mathbb{C}$ of complex numbers etc. Notice that Schmidt does not mention the field K' explicitly).
Let B^e be a subspace of $\mathbb{C}^n$ defined over K. Schmidt first assumes that (1.1) holds. Let $m = n - e$ and $B^{e,\perp} := (B^e)^\perp = \mathrm{Span}\,(Z_1, \ldots, Z_m)$ with $Z_i \in K'^n$, where $\mathrm{Span}\,(T_1, \ldots, T_m)$ denotes the subspace generated by $T_1, \ldots, T_m$. We are going to follow Schmidt's idea in order to construct a vector $W \in K'^n \setminus (B^e)^\perp$ such that

$$B^{e-1} := \mathrm{Span}\,(W, Z_1, \ldots, Z_m)^\perp \tag{4.1}$$

(which is defined over K) has the required properties.
Let $\lambda_i := \Big(1 - \omega_i(A^d, B^e)\Big)^{1/2}$ (for $i = 1, \ldots, f := \min(d,e)$) and choose orthonormal bases $(X_1, \ldots, X_d)$, $(Y_1, \ldots, Y_e)$ of A^d and B^e respectively having $\langle X_i, Y_j\rangle = \delta_{ij}\lambda_i$ (such bases are given by Proposition 2), whence for $i = 1, \ldots, f$ one has $\mathrm{dist}\,(X_i, Y_i) = \omega_i(A^d, B^e)$. Notice that $\langle Y_i, Z_j\rangle = 0$ ($i = 1, \ldots, e$ and $j = 1, \ldots, m$).

The lattice $\Lambda(B^{e,\perp}) \subset E^{pn}$ (constructed from K' and ρ as in §2) has rank pm and determinant $\Delta^m H(B^{e,\perp}) = \Delta^m H(B^e)$ (by (3.1)). Let $\mathfrak{I}_1, \ldots, \mathfrak{I}_{pm}$ be a basis of this lattice and let Π be the set of points $\mathfrak{I} = \sum c_i \mathfrak{I}_i$ with $c_i \in [-1/2, 1/2]$. The set Π has pm-dimensional volume $\Delta^m H(B^e)$ and contains no lattice point of $\Lambda := \Lambda(K'^n)$ but 0. Set $S^* := \mathrm{Span}\,\big(\Lambda(B^{e,\perp})\big) \subset E^{pn}$; it has dimension pm.

4.1 Construction of W

Remember that ρ is defined with respect to K' (not K). For all $Y \in \mathbb{C}^n$ and $Z \in K'^n$, one has

$$\langle Y, Z\rangle = \Big\langle (Y^{[1]}, Y^{[2]}, 0, \ldots, 0), \rho(Z)\Big\rangle + i\Big\langle (-Y^{[2]}, Y^{[1]}, 0, \ldots, 0), \rho(Z)\Big\rangle.$$

$(Y_j)_j$ is an orthonormal basis and each Y_j is orthogonal to $B^{e,\perp}$, hence

$$\mathfrak{Y}_j^1 := (Y_j^{[1]}, Y_j^{[2]}, 0, \ldots, 0) \quad (j = 1, \ldots, h),$$

and

$$\mathfrak{Y}_j^2 := (-Y_j^{[2]}, Y_j^{[1]}, 0, \ldots, 0) \quad (j = 1, \ldots, h),$$

form an orthonormal family of $2h$ vectors of E^{pn} which is orthogonal to S^*. Set $T_h := \mathrm{Span}\,\big(\mathfrak{Y}_1^1, \mathfrak{Y}_1^2, \ldots, \mathfrak{Y}_h^1, \mathfrak{Y}_h^2\big)$; then T_h is a subspace of dimension $2h$. A vector $\mathfrak{X} \in E^{pn}$ can be uniquely written as

$$\mathfrak{X} = \mathfrak{X}^* + \mathfrak{X}_T + \mathfrak{X}_0,$$

with $\mathfrak{X}^* \in S^*$, $\mathfrak{X}_T \in T_h$ and $\mathfrak{X}_0$ orthogonal to S^* and to T_h. The set of all $\mathfrak{X}$ satisfying

- (i) $\mathfrak{X}^* \in \Pi$
- (ii) $|\langle \mathfrak{X}_T, \mathfrak{Y}_j^i\rangle| \le H^{-(y_j-(2y-1)/(ep))}\big(H/H(B^e)\big)^{1/(2h)} \quad (j = 1, \ldots, h\,;\, i = 1, 2)$
- (iii) $||\mathfrak{X}_0|| \le C_{12} H^{(2y-1)/(ep)}$

(where $||\cdot||$ is the norm associated with $\langle\cdot,\cdot\rangle$) is a symmetric convex body which is the product of three symmetric convex bodies from pairwise orthogonal subspaces, hence it has a volume

$$\Delta^m H(B^e) \times \Big(\prod_{j=1}^{h} \big(2H^{-(y_j-(2y-1)/(ep))} \big(H/H(B^e) \big)^{1/(2h)} \big)^2 \Big) \times$$
$$\times \big(C_{12} H^{(2y-1)/(ep)} \big)^{pe-2h} V(pe-2h),$$

where $V(l)$ denotes the volume of the unit ball in E^l. Finally its volume is $\Delta^m 4^h V(pe-2h) C_{12}^{ep-2h} > 2^{pn}\Delta^n$ if C_{12} is large enough. Therefore, by Minkowski's Theorem, there is an $\mathfrak{X} \in \Lambda \setminus \{0\}$ in this set. One may choose W in $\mathcal{O}_{K'}^n$ such that

$$\mathfrak{X} = \rho(W).$$

4.2 Properties of W

One has to establish two properties for W (inequalities (4.2) and (4.3) below) in order to show that B^{e-1} defined by (4.1) has all the required properties. More precisely, $|\langle W, Y_j \rangle|$ and $||V_j||$ have to be controlled (where V_j is the orthogonal projection of $W^{(j)}$ on Span $\left(Z_1^{(j)}, \dots, Z_m^{(j)} \right)^{\perp}$) because these quantities will appear directly in the estimate of the height $H(B^{e-1})$ of B^{e-1}.

For $1 \le j \le h$ one has

$$\begin{aligned} |\langle Y_j, W \rangle| &= |\langle \mathfrak{Y}_j^1, \rho(W) \rangle + i \langle \mathfrak{Y}_j^2, \rho(W) \rangle| = |\langle \mathfrak{Y}_j^1, \mathfrak{X} \rangle + i \langle \mathfrak{Y}_j^2, \mathfrak{X} \rangle| \\ &\le |\langle \mathfrak{Y}_j^1, \mathfrak{X}_T \rangle| + |\langle \mathfrak{Y}_j^2, \mathfrak{X}_T \rangle| \\ &\le 2H^{-(y_j-(2y-1)/(ep))} \big(H/H(B^e) \big)^{1/(2h)}, \end{aligned} \tag{4.2}$$

by (ii).
Also notice that (ii) and (iii) together imply

$$||\mathfrak{X} - \mathfrak{X}^*|| \le C_{14} H^{(2y-1)/(ep)},$$

because by assumption $y_i \ge 1/(2h)$ (which implies that $H^{-(y_j-(2y-1)/(ep))} \times \big(H/H(B^e) \big)^{1/(2h)}$ is bounded from above by $H^{(2y-1)/(ep)}$).
For $1 \le j \le p$, write $W^{(j)} = U_j + V_j$ with $U_j \in$ Span $\left(Z_1^{(j)}, \dots, Z_m^{(j)} \right)$ and V_j orthogonal to Span $\left(Z_1^{(j)}, \dots, Z_m^{(j)} \right)$ (here Schmidt's arguments do not work exactly as he says, and this is what motivates us to introduce V_j, $j = 1, \dots, p$. See Remarks 4 and 5 for more details).
Now if σ_j is real, then $W^{(j)}, Z_1^{(j)}, \dots, Z_m^{(j)} \in \mathbb{R}^n$, and this implies that $U_j, V_j \in \mathbb{R}^n$. Set $\mathfrak{W}_j := (0, \dots, \underbrace{V_j}_{j-\text{th block}}, \dots, 0)$. Then $\mathfrak{W}_j$ is orthogonal to S^*, thus to $\mathfrak{X}^*$, and to $\mathfrak{X} - \mathfrak{W}_j$ (by definition of V_j and U_j). From this one can deduce that $||\mathfrak{W}_j||^2 = |\langle \mathfrak{W}_j, \mathfrak{X} \rangle| = |\langle \mathfrak{W}_j, \mathfrak{X} - \mathfrak{X}^* \rangle| \le ||\mathfrak{W}_j|| \times C_{14} H^{(2y-1)/(ep)}$. Thus

$$||V_j|| = ||\mathfrak{W}_j|| \le C_{14} H^{(2y-1)/(ep)}.$$

If σ_j and σ_{j+1} are complex conjugate, then

$$\mathfrak{W}_j^1 := (0, \dots, 0, \underbrace{V_j^{[1]}, V_j^{[2]}}_{\text{blocks } j \text{ and } j+1}, 0, \dots, 0) \text{ and } \mathfrak{W}_j^2 := (0, \dots, 0, \underbrace{-V_j^{[2]}, V_j^{[1]}}_{\text{blocks } j \text{ and } j+1}, 0, \dots, 0)$$

are orthogonal to S^*, and in particular to $\mathfrak{X}^*$. Then, one has

$$\left| \left\langle W^{[j]}, V_j^{[1]} \right\rangle + \left\langle W^{[j+1]}, V_j^{[2]} \right\rangle \right| = |\langle \mathfrak{X}, \mathfrak{W}_j^1 \rangle| = |\langle \mathfrak{X} - \mathfrak{X}^*, \mathfrak{W}_j^1 \rangle| \le C_{14} H^{(2y-1)/(ep)} ||\mathfrak{W}_j^1||,$$
$$\left| \left\langle W^{[j]}, V_j^{[2]} \right\rangle - \left\langle W^{[j+1]}, V_j^{[1]} \right\rangle \right| = |\langle \mathfrak{X}, \mathfrak{W}_j^2 \rangle| = |\langle \mathfrak{X} - \mathfrak{X}^*, \mathfrak{W}_j^2 \rangle| \le C_{14} H^{(2y-1)/(ep)} ||\mathfrak{W}_j^2||,$$

and since

$$\Big|\Big\langle W^{(j)},V_j\Big\rangle\Big| = \Big|\Big\langle W^{[j]},V_j^{[1]}\Big\rangle + \Big\langle W^{[j+1]},V_j^{[2]}\Big\rangle + i\big(\Big\langle W^{[j]},V_j^{[2]}\Big\rangle - \Big\langle W^{[j+1]},V_j^{[1]}\Big\rangle\big)\Big|,$$

one may conclude that

$$||V_j||^2 = \Big|\Big\langle W^{(j)},V_j\Big\rangle\Big| \le 2C_{14}H^{(2y-1)/(ep)}||V_j||,$$

and therefore

$$||V_j|| \le 2C_{14}H^{(2y-1)/(ep)}. \tag{4.3}$$

This inequality is satisfied for $j=1,\dots,p$.

4.3 Definition and properties of B^{e-1}

As announced, let B^{e-1} be defined by (4.1). Now one has to show that $H(B^{e-1}) \le H'$. We follow Schmidt's arguments. Let $\mathfrak{a}$ be the fractional ideal of K' generated by the $\binom{n}{m}$ determinants of all $m\times m$-submatrices of the matrix with row vectors $Z_1,\dots,Z_m$, and let $\mathfrak{b}$ be the ideal of K' generated by the $\binom{n}{m+1}$ determinants of all $(m+1)\times(m+1)$-submatrices of the matrix with row vectors $W,Z_1,\dots,Z_m$. Since all coordinates of W are in $\mathcal{O}_{K'}$ one has $\mathfrak{b}\subset\mathfrak{a}$ (for one can use for every $(m+1)\times(m+1)$-determinant the Laplace expansion along the first row). Thus $N(\mathfrak{b})\ge N(\mathfrak{a})$. Moreover, $H(B^e) = H(B^{e,\perp}) = N(\mathfrak{a})^{-1}\prod_{i=1}^p ||Z_1^{(i)}\wedge\cdots\wedge Z_m^{(i)}||$. Then, one has

$$\begin{aligned}
H(B^{e-1}) = H((B^{e-1})^{\perp}) &= N(\mathfrak{b})^{-1}\prod_{i=1}^{p}||W^{(i)}\wedge Z_1^{(i)}\wedge\cdots\wedge Z_m^{(i)}||\\
&= N(\mathfrak{b})^{-1}\prod_{i=1}^{p}||V_i||\,||Z_1^{(i)}\wedge\cdots\wedge Z_m^{(i)}|| \le H(B^e)\prod_{i=1}^{p}||V_i||\\
&\le C_5H(B^e)H^{(2y-1)/e} \le C_5H^{(e+2y-1)/e} = H',
\end{aligned} \tag{4.4}$$

for a sufficiently large C_5.

Now one has to bound $H(B^{e-1})\omega_i^2(A^d,B^{e-1})$ from above (for $i=1,\dots,h$). Since $\langle Y_i,Z_j\rangle = 0$ $(j=1,\dots,m)$ and $||Y_i||=1$, one has

$$\begin{aligned}
&\operatorname{dist}\big(Y_i,(B^{e-1})^{\perp}\big)^2 =\\
&= ||Y_i\wedge W\wedge Z_1\wedge\cdots\wedge Z_m||^2||Y_i||^{-2}||W\wedge Z_1\wedge\cdots\wedge Z_m||^{-2}\\
&= \Big(||Y_i||^2||W\wedge Z_1\wedge\cdots\wedge Z_m||^2 - |\langle Y_i,W\rangle|^2||Z_1\wedge\cdots\wedge Z_m||^2\Big)||W\wedge Z_1\wedge\cdots\wedge Z_m||^{-2}\\
&= 1-|\langle Y_i,W\rangle|^2||Z_1\wedge\cdots\wedge Z_m||^2||W\wedge Z_1\wedge\cdots\wedge Z_m||^{-2},
\end{aligned}$$

the first equality is obtained by the special case of Proposition 3 and the second one is obtained by using Laplace's expansion twice (first for $D^2(Y_i,W,Z_1,\dots,Z_m)$ defined at the beginning of §2 along the first column, then for the second non-zero determinant obtained along the first row).
Using Eq. (2.2) one finds

$$\operatorname{dist}\big(Y_i,B^{e-1}\big) = |\langle Y_i,W\rangle|\,||Z_1\wedge\cdots\wedge Z_m||\,||W\wedge Z_1\wedge\cdots\wedge Z_m||^{-1},$$

hence

$$||W\wedge Z_1\wedge\cdots\wedge Z_m||^2\operatorname{dist}\big(Y_i,B^{e-1}\big)^2 = |\langle Y_i,W\rangle|^2||Z_1\wedge\cdots\wedge Z_m||^2.$$

Finally

$$\begin{aligned}
&H(B^{e-1})\ \mathrm{dist}\left(Y_i, B^{e-1}\right)^2 \\
&= N(\mathfrak{b})^{-1}\Big(\prod_{j=3}^{p} ||W^{(j)} \wedge Z_1^{(j)} \wedge \cdots \wedge Z_m^{(j)}||\Big) ||W \wedge Z_1 \wedge \cdots \wedge Z_m||^2\ \mathrm{dist}\left(Y_i, B^{e-1}\right)^2 \\
&= N(\mathfrak{b})^{-1}\Big(\prod_{j=3}^{p} ||W^{(j)} \wedge Z_1^{(j)} \wedge \cdots \wedge Z_m^{(j)}||\Big) ||Z_1 \wedge \cdots \wedge Z_m||^2 |\left\langle Y_i, W\right\rangle|^2 \\
&\le C_{15} N(\mathfrak{a})^{-1}\Big(\prod_{j=1}^{p} ||Z_1^{(j)} \wedge \cdots \wedge Z_m^{(j)}||\Big)\Big(\prod_{j=3}^{p} ||V_j||\Big) H^{-2(y_i-(2y-1)/(ep))}\big(H/H(B^e)\big)^{1/h} \\
&\le C_{16} H(B^e) H^{(p-2)(2y-1)/(ep)-2[y_i-(2y-1)/(ep)]} H/H(B^e) \\
&\le C_{16} H^{(e+2y-1)/e-2y_i}.
\end{aligned}$$

The first inequality follows from (4.2) and the second one from (4.3). Then, there is a vector $R_i \in B^{e-1} \setminus \{0\}$ having

$$H(B^{e-1})\ \mathrm{dist}\,(Y_i, R_i)^2 \le C_{16} H^{-(2y_i'-1)(2y+e-1)/e},$$

since by definition $y_i' := (y_i e)/(qy + e - 1)$.
By assumption, one has

$$H(B^e)\ \mathrm{dist}\,(X_i, Y_i)^2 \le c^2 H^{-2y_i+1},$$

thus, by (4.4) :

$$H(B^{e-1})\ \mathrm{dist}\,(X_i, Y_i)^2 \le c^2 C_5 H^{-2y_i+1+(2y-1)/e} = c^2 C_5 H^{-(2y_i'-1)(2y+e-1)/e}.$$

These inequalities and the triangle inequality provide

$$H(B^{e-1})\ \mathrm{dist}\,(X_i, R_i)^2 \le C_{17} c^2 H^{-2y_i+1+(2y-1)/e} = C_{17} c^2 H^{-(2y_i'-1)(2y+e-1)/e} \quad (i = 1, \ldots, h),$$

and so [10, Theorem 7] (with $\delta = 1$ for instance, since $(X_j)_j$ is an orthonormal family) yields

$$H(B^{e-1})\omega_i^2(A^d, B^{e-1}) \le c^2 C_6 H^{-(2y_i'-1)(2y+e-1)/e} \quad (i = 1, \ldots, h),$$

for C_6 large enough. This completes the first part of the proof.

4.4 Proof of the second part of the theorem

Suppose now that (1.3) holds. We follow Schmidt's proof and define $H_1 \ge 1$ by $H' = C_9 H H_1^{2h/e}$ (C_9 will be specified later). For the construction of W, replace (i), (ii), (iii) by

(i') $\mathfrak{X}^* \in \Pi$

(ii') $|\left\langle \mathfrak{X}_T, \mathfrak{Y}_j^i \right\rangle| \le H_1^{-(1-2h/(ep))} \quad (j = 1, \ldots, h\,;\, i = 1, 2)$

(iii') $||\mathfrak{X}_0|| \le C_{18} H_1^{2h/(ep)}$.

These equations define a symmetric convex body of volume

$$\begin{aligned}
&\Delta^m H(B^e) \times (2H_1^{-(1-2h/(ep))})^{2h} \times (C_{18} H_1^{2h/(ep)})^{ep-2h} V(ep-2h) \\
&= \Delta^m H(B^e) 4^h C_{18}^{ep-2h} V(ep-2h) \\
&> 2^{pn}\Delta^n,
\end{aligned}$$

for C_{18} large enough. By Minkowski's Theorem there is an $\mathfrak{X} \in \Lambda \setminus \{0\}$ in this set. Let W be in $\mathcal{O}_{K'}^n$ such that $\mathfrak{X} = \rho(W)$.
Equation (4.2) is replaced by

$$|\langle W, Y_j \rangle| \leq 2H_1^{-(1-2h/(ep))}.$$

One also has

$$||\mathfrak{X} - \mathfrak{X}^*|| \leq C_{19} H_1^{2h/(ep)},$$

thus (4.3) becomes

$$||V_j|| \leq 2C_{19} H_1^{2h/(ep)}.$$

Computing $H(B^{e-1})$ one finds

$$H(B^{e-1}) \leq H(B^e) \prod_{j=1}^{p} ||V_j|| \leq CH(B^e) H_1^{2h/e}.$$

Now set $C_9 := C$, which implies $H(B^{e-1}) \leq H'$. With these new estimates one finds

$$\begin{aligned} H(B^{e-1}) \text{ dist} \left(Y_i, B^{e-1}\right)^2 &\leq C_{20} H(B^e) \times H_1^{(p-2)2h/(ep)} \times H_1^{-2(1-2h/(ep))} \\ &\leq C_{21} H H_1^{2h/e-2} = C_{21} H H_1^{2h/e(1-e/h)} \\ &= C_{10} H^{e/h} H'^{1-e/h} = C_{10} H^{2y_0'} H'^{-(2y_0'-1)}. \end{aligned}$$

Now, note that (1.3) implies that $Y_i = \pm X_i$ (for $i = 1, \dots, h$) and so $Y_1, \dots, Y_h$ form an orthonormal subset of A^d; [10, Theorem 7] (with $\delta = 1$) yields the expected result.

□

Remark 4. We give here more details in case (b) about the reasons which lead us to introduce subspaces V_j and the decomposition of vectors $W^{(j)}$ (see Subsection 4.2). In his paper [10, p. 455], Schmidt writes (with his notation for the orthogonal complement and for $Z_j \in K^n$) $W = U + V$, where $U \in B^{e\perp} = \text{Span}\,(Z_1, \dots, Z_m)$ and $V \in B^e$, in order to have the decomposition $W^{(j)} = U^{(j)} + V^{(j)}$ with $U^{(j)} \in (B^e)^{(j)\perp} = \text{Span}\left(Z_1^{(j)}, \dots, Z_m^{(j)}\right)$ and $V^{(j)} \in (B^e)^{(j)}$.
Here a problem arises : in the complex case, which orthogonal complement does the symbol $\perp$ denote ? With our notation, suppose that one considers $(B^e)^{\varphi,\perp}$. Then, it is not always true that we have $G^n = B^e \oplus (B^e)^{\varphi,\perp}$ and so the decomposition $W = U + V$ may not be considered. (This is the first but not the only one problematic point: for example with this definition it seems that we could not obtain inequality (15) of Schmidt, because just before in his text the notation $W^{(j)} R^{(j)}$ would refer to $\varphi(W^{(j)}, R^{(j)})$ and not to the inner product $\langle W^{(j)}, R^{(j)} \rangle$).
Suppose now that $(B^e)^\perp$ denotes the orthogonal complement for the inner product $\langle \cdot, \cdot \rangle$. Here, the decomposition $W = U + V$ may be considered. However it rises another problem : $(B^e)^\perp$ is no longer defined over K, but over K', and we could not apply the embedding $\sigma_j : K \to \mathbb{C}$ to U. In fact we also have a problem to define $V^{(j)}$ although B^e is defined over K, because we do not have necessarily $V \in K^n$ (the decomposition $(0, 1) = \lambda(1, \alpha) + \lambda(-1, (\overline{\alpha})^{-1})$ with $\lambda = 1/(\alpha + (\overline{\alpha})^{-1})$ and $\alpha \in K \setminus K'$ provides a simple counter-example in dimension two). Furthermore, even if $K = K'$ and that $U^{(j)}$ and $V^{(j)}$ could be considered, it is not true that it implies $\langle U^{(j)}, V^{(j)} \rangle = 0$, because of the complex conjugation in the inner product. This brings problems to apply Schmidt's arguments.

Remark 5. In case (a), it makes sense to considerer the decomposition $W^{(j)} = U^{(j)} + V^{(j)}$ of Schmidt, but if K is not totally real and if σ_j is a non-real embedding, with Schmidt's argument it seems that one can only obtain

$$|\varphi(V^{(j)}, V^{(j)})| \leq 2C_{14}H^{(y-1)/(ep)}||V^{(j)}||,$$

instead of inequality (15) of [10] (which corresponds to inequality (4.3) in this paper). To solve this problem, it suffices to apply the argument used for the complex case. It is simpler in case (a) because we have $K' = K$, and B^e is the subspace defined by Schmidt in [9] (defined over K). If we replace the decomposition $U^{(j)} = W^{(j)} + U^{(j)}$ with the decomposition $W^{(j)} = U_j + V_j$ if σ_j is not real, with $U_j \in \text{ Span}\left(Z_1^{(j)}, \ldots, Z_m^{(j)}\right)$ and V_j orthogonal to $\text{ Span}\left(Z_1^{(j)}, \ldots, Z_m^{(j)}\right)$, as for (4.3) we obtain

$$||V_j|| \leq 2C_{14}H^{(y-1)/(ep)}.$$

This inequality is slightly different from (4.3) because in case (a) the symmetric convex body defined by (i), (ii) and (iii) at the beginning of the proof is not the same. Now, note that in Section 4.3 we use inequalities (4.2) and (4.3) but not directly the fact that $K \subset \mathbb{R}$ or not, so working with V_j rather than $V^{(j)}$ we may follow Schmidt's arguments to complete the proof of case (a).

5 Exponents of Diophantine Approximation

Let $n \geq 1$ and $K \subset \mathbb{C}$ be a number field. We recall that one distinguishes between two cases (a) and (b).
In case (a), K is real, G^{n+1} denotes Euclidean space $\mathbb{R}^{n+1}$, and $q = 1$.
In case (b), K is complex non real, G^{n+1} denotes unitary $\mathbb{C}^{n+1}$, and $q = 2$.

Definition 6. Let $\mathbf{u} \in G^{n+1} \setminus \{0\}$. For each $j = 0, \ldots, n-1$, we denote by $\omega_{j,K}(\mathbf{u})$ (resp. $\widehat{\omega}_{j,K}(\mathbf{u})$) the supremum of all real numbers ω such that, for arbitrarily large values of Q (resp. for all sufficiently large values of Q), there exists a vector subspace S of G^{n+1}, defined over K, of dimension $j+1$, with

$$H(S) \leq Q \quad \text{and} \quad H(S)\omega_1^q(\mathbf{u}, S) \leq Q^{-\omega}.$$

Laurent introduces this family of exponents in [5] in the case $K = \mathbb{Q}$. He gives a series of inequalities (which may be proved using Schmidt's results [10]) relating these exponents (cf for example [8, Theorem 2.2], and compare to Theorem 8 below), and gives also a description of the full spectrum of the $2n$ exponents $(\omega_{0,\mathbb{Q}}(\mathbf{u}), \ldots, \omega_{n-1,\mathbb{Q}}(\mathbf{u}), \ldots, \widehat{\omega}_{0,\mathbb{Q}}(\mathbf{u}), \ldots, \widehat{\omega}_{n-1,\mathbb{Q}}(\mathbf{u}))$ for $n = 2$. In [8], Roy gives a description of the full spectrum of the n exponents $(\omega_{0,\mathbb{Q}}(\mathbf{u}), \ldots, \omega_{n-1,\mathbb{Q}}(\mathbf{u}))$ for all $n \geq 1$, proving that the inequalities of Laurent describe completely this spectrum (cf Theorem 2.3 of [8]).

For an arbitrary number field K, the aforementioned inequalities can be generalized: this is Theorem 8, which is the main application of this section.

Theorem 7. *Let* $\mathbf{u} \in G^{n+1} \setminus \{0\}$ *and let* j, $0 \leq j \leq n-1$. *Then*

$$\omega_{j,K}(\mathbf{u}) \geq \widehat{\omega}_{j,K}(\mathbf{u}) \geq \frac{j+1}{n-j} \quad (0 \leq j \leq n-1). \tag{5.1}$$

Proof[of Theorem 7]
By [10, Theorem 13] in the case $d = 1$, there exists a constant $c > 0$ which depends only of n and K such that for all $Q \geq 1$, there is a subspace S, defined over K and of dimension $j + 1$, satisfying

$$H(S) \leq Q \quad \text{and} \quad H(S)\omega_1^q(\mathbf{u}, S) \leq cQ^{-(j+1)/(n-j)}.$$

Theorem 7 follows immediately. Note that in the proof of Theorem 13, Schmidt uses his going-down Theorem.

□

Theorem 8. *Let $n \in \mathbb{N}^*$ and $\mathbf{u} \in G^{n+1}$. Then we have $\omega_{0,K}(\mathbf{u}) \geq \frac{1}{n}$ and*

$$\frac{j\omega_{j,K}(\mathbf{u})}{\omega_{j,K}(\mathbf{u}) + j + 1} \leq \omega_{j-1,K}(\mathbf{u}) \leq \frac{(n-j)\omega_{j,K}(\mathbf{u}) - 1}{n - j + 1} \quad (1 \leq j \leq n-1), \tag{5.2}$$

with the convention that the left-most ratio is equal to j if $\omega_{j,K}(\mathbf{u}) = \infty$.

The right inequality in (5.2) follows from Schmidt's going-up Theorem [10, Theorem 9], while the left inequality follows from the going-down Theorem. In his paper [5], Laurent introduces the exponents $\omega_{j,\mathbb{Q}}(\mathbf{u})$ and notes that each inequalities in (5.2) is best possible (because they allow to find Khinchine's transference inequalities, which are best possible). For $\mathbf{u}$ with $\mathbb{Q}-$linearly independent coordinates and $K = \mathbb{Q}$, Laurent gives an independent proof of the right inequality of (5.2) in [5], and both inequalities are proved by Bugeaud and Laurent in [1]. Roy proves that for $K = \mathbb{Q}$, the inequalities of Theorem 8 describe the set of all possible values of the n-tuples $(\omega_{0,\mathbb{Q}}(\mathbf{u}), \ldots, \omega_{n-1,\mathbb{Q}}(\mathbf{u}))$ (cf [8, Theorem 2.3]). It would be of interest to know if his theorem remains true or not for an arbitrary number field K.

Proof

We show first the left inequality in (5.2).

Set $A^1 := \text{Span}(\mathbf{u})$, the line spanned by $\mathbf{u}$ in G^{n+1}. Let j, $1 \leq j \leq n-1$. If $\omega_{j,K}(\mathbf{u}) = \infty$ let $y_1 > q^{-1}$. Otherwise let $y_1 = q^{-1}(\omega_{j,K}(\mathbf{u}) + 1) - \varepsilon$, where $\varepsilon > 0$ is very small; then $y_1 > q^{-1}$ because Theorem 7 yields $\omega_{j,K}(\mathbf{u}) > 0$. Then, there exist arbitrarily large values of Q for which there exists a subspace S of G^{n+1} of dimension $j + 1$ $(1 \leq j \leq n-1)$, such that

$$H(S) \leq Q \quad \text{and} \quad H(S)\omega_1^q(A^1, S) \leq Q^{-(qy_1-1)}. \tag{5.3}$$

Set $y_1' := y_1(j+1)/(qy_1+j)$. The conditions of the going-down Theorem are fulfilled with A^1 and $B^e := S$ (one has $d = i = h = 1$, $e = j+1$ and we choose $c = 1$). This gives a subspace $S' \subset S$ of height $H(S') \leq Q'$ with $Q' := C_5 Q^{\frac{qy_1+j}{j+1}}$, of dimension j, defined over K, having

$$H(S') \leq Q' \quad \text{and} \quad H(S')\omega_1^q(\mathbf{u}, S') \leq C_7 Q'^{-(qy_1'-1)}. \tag{5.4}$$

Since Equation (5.3) holds for arbitrarily larges values of Q, (5.4) holds also for arbitrarily large values of Q', and one deduces that

$$\omega_{j-1,K}(\mathbf{u}) \geq qy_1' - 1.$$

Letting y_1 tends to $q^{-1}(\omega_{j,K}(\mathbf{u}) + 1)$ if $\omega_{j,K}(\mathbf{u}) < \infty$, and to $+\infty$ otherwise, we deduce the left side of the inequality in (5.2).

Only a scheme of proof is given for the right inequality.
Set again $A^1 := \mathrm{Span}(\mathbf{u})$ and let j, $1 \le j \le n-1$. We can suppose that for any subspace S of dimension $j+1$, defined over K, we have $\omega_1^q(A^1, S) > 0$ (otherwise it means that $\omega_{j,K}(\mathbf{u}) = \infty$, and the right inequality in (5.2) is obvious). Let y_1, $0 < qy_1 < \omega_{j-1,K}(\mathbf{u})$. There exist arbitrarily large values $Q \ge 1$ and corresponding subspaces S of dimension j defined over K, having

$$H(S) \le Q \quad \text{and} \quad H(S)\omega_1^q(A^1, S) \le Q^{-qy_1}. \tag{5.5}$$

Now, we use Schmidt's going-up Theorem ([10, Theorem 8]) with parameters $n+1$, $x_1 = 1/q$, y_1, $d = t = i = 1$, $c = 1$, and $B^e := S$ (note that in our context we have $\psi_1(A^1, B^e) = \omega_1(A^1, B^e)$ for all B^e of dimension $e < n$). This gives us a subspace $S' \supseteq S$ of dimension $j+1$, defined over K, having

$$H(S') \le Q' \quad \text{and} \quad 0 < H(S')^{(n-j+1)/(n-j)}\omega_1^q(A^1, S') \le C_4 Q'^{-qy_1(n-j+1)/(n-j)}, \tag{5.6}$$

where $Q' := C_3 Q^{(n-j)/(n-j+1)}$ and $C_3, C_4 > 0$ depend of n, K and y_1 only.
Now, since Equation (5.5) holds for arbitrarily large values Q, so does (5.6) for arbitrarily large values Q'. This implies that there are infinitely many subspaces S' which satisfy (5.6). In particular, $H(S')$ is not bounded from above.
To conclude, it suffices to remark that (5.6) implies

$$0 < H(S')^{(n-j+1)/(n-j)}\omega_1^q(A^1, S') \le C_4 H(S')^{-qy_1(n-j+1)/(n-j)}.$$

Then multiplying each side of the inequality by $H(S')^{1-(n-j+1)/(n-j)}$ and writing $Q'' = H(S')$, we find that for arbitrarily large values of Q'' there exists S' of dimension $j+1$ defined over K such that

$$H(S') \le Q'' \quad \text{and} \quad H(S')\omega_1^q(A^1, S') \le C_4 Q''^{-(qy_1+1)(n-j+1)/(n-j)+1}. \tag{5.7}$$

This shows that

$$\omega_{j,K}(\mathbf{u}) \ge (qy_1 + 1)(n-j+1)/(n-j) - 1,$$

which gives

$$qy_1 \le \frac{(n-j)\omega_{j,K}(\mathbf{u}) - 1}{n-j+1},$$

and letting qy_1 tend to $\omega_{j-1,K}(\mathbf{u})$ we find the desired result.

□

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