A CLASS OF MAXIMALLY SINGULAR SETS FOR RATIONAL APPROXIMATION

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Abstract. We say that a subset of $\mathbb{P}^n(\mathbb{R})$ is maximally singular if its contains points with $\mathbb{Q}$-linearly independent homogenous coordinates whose uniform exponent of simultaneous rational approximation is equal to 1, the maximal possible value. In this paper, we give a criterion which provides many such sets including Grassmannians. We also recover a result of the author and Roy about a class of quadratic hypersurfaces.

1. Introduction

A basic problem in Diophantine approximation is given a subset Z of $\mathbb{P}^n(\mathbb{R})$ to compute

$$\hat{\lambda}(Z) := \sup\{\hat{\lambda}(\xi)\,;\, \xi \in Z^{\mathrm{li}}\} \in [1/n, 1],$$

where Z^{li} stands for the set of elements of Z having representatives with $\mathbb{Q}$-linearly independent coordinates in $\mathbb{R}^{n+1}$, and $\widehat{\lambda}(\xi)$ is the so-called uniform exponent of simultaneous rational approximation to ξ (see Section 2 for the definition). We say that Z is *maximally singular* if $\widehat{\lambda}(Z) = 1$. Diophantine approximation in a projective setting is not a new topic, see for example [2], or [5, 4] for more recent results.

A famous example is the Veronese curve $Z = \{(1 : \xi : \xi^2 : \cdots : \xi^n)\,;\, \xi \in \mathbb{R}\} \subseteq \mathbb{P}^n(\mathbb{R})$ for which the computation of $\widehat{\lambda}(Z)$, open for $n \geq 3$, would have implications on problems of algebraic approximation (see [3]). The case $n = 2$ is treated in [7]. More generally, Roy computes in [8] this exponent for conics in $\mathbb{P}^2(\mathbb{R})$. These results are extended by the author and Roy in [6] to the case of quadratic hypersurface Z of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$. We show that, if Z^{li} is not empty, then the exponent $\widehat{\lambda}(Z)$ is completely determined by n and the Witt index of the quadratic form defining Z [6, Theorem 1.1]. In particular, if this index is at least 2, then $\widehat{\lambda}(Z) = 1$ and there are uncountably many $\xi \in Z^{\mathrm{li}}$ such that $\widehat{\lambda}(\xi) = 1$.

Our main result presented in Section 2 gives a geometric condition under which a closed set $Z \subseteq \mathbb{P}^n(\mathbb{R})$ (for the topology induced by the projective distance) satisfies $\widehat{\lambda}(\xi) = 1$ for an uncountable set of points $\xi \in Z^{\mathrm{li}}$, and thus $\widehat{\lambda}(Z) = 1$. It allows us to recover the above mentionned result of [6] concerning quadratic hypersurfaces of Witt index ≥ 2 (see Section 6). It also admits the following consequence.

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Theorem 1.1. *Let k, n, N be positive integers with $N \geq 2$, let $\varphi : (\mathbb{R}^n)^k \to \mathbb{R}^{N+1}$ be a k-linear map defined over $\mathbb{Q}$ such that $\varphi((\mathbb{Q}^n)^k)$ spans the whole space $\mathbb{R}^{N+1}$ over $\mathbb{R}$, let Z denote the topological closure in $\mathbb{P}^N(\mathbb{R})$ of the projectivization of $\varphi((\mathbb{R}^n)^k)$, then there are uncountably many $\xi \in Z^{li}$ such that $\widehat{\lambda}(\xi) = 1$, and so $\widehat{\lambda}(Z) = 1$.*

Examples of such sets Z are the Grassmannians $G_{k,n}$ of k-dimensional subspaces of $\mathbb{R}^n$ as well as the subset of $\mathbb{R}[x_1, \dots, x_n]_k$ consisting of homogenous polynomials of degree k in n variables which factor as a product of k linear forms over $\mathbb{R}$ (see Section 5 for details). Note that, in general, $\varphi((\mathbb{R}^n)^k)$ need not be a closed subset of $\mathbb{R}^{N+1}$ as an example of Bernau and Wojciechowski [1] shows. So its projectivization may not be closed.

2. Main result and notation

Let n be an integer ≥ 1. For each set $Z \subseteq \mathbb{P}^n(\mathbb{R})$ (resp. $S \subseteq \mathbb{R}^{n+1}$) we write $Z(\mathbb{Q}) := Z \cap \mathbb{P}^n(\mathbb{Q})$ (resp. $S(\mathbb{Q}) = S \cap \mathbb{Q}^{n+1}$). We denote by $[\mathbf{x}]$ the class in $\mathbb{P}^n(\mathbb{R})$ of a non-zero point $\mathbf{x}$ of $\mathbb{R}^{n+1}$. Given non-zero points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$, we recall that the *projective distance* between $[\mathbf{x}]$ and $[\mathbf{y}]$ is

$$\mathrm{dist}([\mathbf{x}], [\mathbf{y}]) := \mathrm{dist}(\mathbf{x}, \mathbf{y}) := \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \, \|\mathbf{y}\|}.$$

Here, $\|\cdot\|$ denotes the Euclidean norm. Let $\xi \in \mathbb{P}^n(\mathbb{R})$ and let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ be a representative of ξ so that $\xi = [\boldsymbol{\xi}]$. Following the notation of [6], we set

$$D_\xi(\mathbf{x}) := \frac{\|\mathbf{x} \wedge \boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|} = \|\mathbf{x}\| \mathrm{dist}(\xi, [\mathbf{x}])$$

for each non-zero $\mathbf{x} \in \mathbb{Z}^{n+1}$. Then, for each $X \geq 1$, we define

$$\mathcal{D}_\xi(X) := \min \left\{ D_\xi(\mathbf{x}) \, ; \, \mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\} \text{ and } \|\mathbf{x}\| \leq X \right\}.$$

The exponent of uniform approximation $\widehat{\lambda}(\xi)$ alluded to in the introduction is defined as the supremum of all $\lambda \in \mathbb{R}$ such that $\mathcal{D}_\xi(X) \leq X^{-\lambda}$ for each sufficiently large X.

Our main result is the following.

Theorem 2.1. *Let $n \geq 2$ and let Z be a closed subset of $\mathbb{P}^n(\mathbb{R})$ (for the topology induced by the projective distance). Suppose that there exists a non-empty set $Z' \subseteq Z(\mathbb{Q})$ with the following property. For each proper projective subspace H of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ there is a function $s_H : Z' \to \mathbb{N}$ such that, for each $x \in Z'$*

(i) *if $s_H(x) = 0$, then $x \notin H$;*
(ii) *if $s_H(x) \geq 1$, then there exists a projective line L defined over $\mathbb{Q}$ such that*

$$x \in L(\mathbb{Q}) \subseteq Z' \quad \text{and} \quad \#\{y \in L(\mathbb{Q}) \, ; \, s_H(y) \geq s_H(x)\} < \infty.$$

Finally let $\varphi : [1, \infty) \to (0, 1]$ be a monotonically decreasing function with $\lim_{X\to\infty} \varphi(X) = 0$ and $\lim_{X\to\infty} X\varphi(X) = \infty$. Then there are uncountably many $\xi \in Z^{\mathrm{li}}$ such that $\mathcal{D}_\xi(X) \leq \varphi(X)$ for all sufficiently large X.

Choosing $\varphi = \log(3X)/X$ for $X \geq 1$, we derive the following consequence.

Corollary 2.2. *With the hypotheses and notation of the previous theorem, there are uncountably many $\xi \in Z^{\mathrm{li}}$ such that $\widehat{\lambda}(\xi) = 1$, and so $\widehat{\lambda}(Z) = 1$.*

3. Proof of Theorem 2.1

Assume that $Z \subseteq \mathbb{P}^n(\mathbb{R})$ and $Z' \subseteq Z(\mathbb{Q})$ satisfy the hypotheses of Theorem 2.1.

Lemma 3.1. *Let H be a proper projective subspace of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ and let $s_H : Z' \to \mathbb{N}$ be a function satisfying Conditions* (i)-(ii) *of Theorem 2.1 for this choice of H. Then for any non-zero integer point $\mathbf{x} \in \mathbb{Z}^{n+1}$ such that $[\mathbf{x}] \in Z' \cap H$ we have $s_H([\mathbf{x}]) \geq 1$ and there exist infinitely many non-zero integer points $\mathbf{y}$ with $[\mathbf{y}] \in Z'$ satisfying*

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) \leq \frac{C}{\|\mathbf{y}\|} \quad \text{and} \quad s_H([\mathbf{y}]) < s_H([\mathbf{x}]), \tag{1}$$

for a constant $C = C(\mathbf{x}, H) > 0$ independent of $\mathbf{y}$.

Proof. Since $[\mathbf{x}] \in H$, Condition (i) implies that $s_H([\mathbf{x}]) \geq 1$. By Condition (ii) of Theorem 2.1, there exists a non-zero integer point $\mathbf{z}$ such that $\mathbb{P}(\langle \mathbf{x}, \mathbf{z} \rangle_{\mathbb{Q}}) \subseteq Z'$ and $s_H([\mathbf{z} + b\mathbf{x}]) \leq s_H([\mathbf{x}]) - 1$ for all but finitely many $b \in \mathbb{Z}$. Putting $\mathbf{y} := \mathbf{z} + b\mathbf{x}$ we find

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x} \wedge \mathbf{z}\|}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

with a numerator that is independent of the choice of b. □

Proof of Theorem 2.1. This is similar in many aspects to the proof of [6, Proposition 10.4]. Starting with a non-zero integer point $\mathbf{x}_1$ with $[\mathbf{x}_1] \in Z'$, we construct recursively a sequence $(\mathbf{x}_i)_{i \geq 1}$ of non-zero points of $\mathbb{Z}^{n+1}$ which satisfies the following properties. When $i \geq 1$ we have

(a) $[\mathbf{x}_{i+1}] \in Z'$;
(b) $\|\mathbf{x}_{i+1}\| > \|\mathbf{x}_i\|$;
(c) $s_H([\mathbf{x}_{i+1}]) < s_H([\mathbf{x}_i])$ for the subspace $H = H_i$ of $\mathbb{P}^n(\mathbb{R})$ given by

$$H_i := \mathbb{P}\Big(\langle \mathbf{x}_j, \dots, \mathbf{x}_i \rangle_{\mathbb{R}}\Big),$$

where j is the smallest index ≥ 1 such that H_i is a proper subspace of $\mathbb{P}^n(\mathbb{R})$.

When $i \geq 2$, we further ask that

(d) $\operatorname{dist}(\mathbf{x}_{i+1}, \mathbf{x}_i) \leq \dfrac{C}{\|\mathbf{x}_{i+1}\|} \leq \dfrac{1}{3} \min\left\{ \dfrac{2\varphi(\|\mathbf{x}_{i+1}\|)}{\|\mathbf{x}_i\|}, \operatorname{dist}(\mathbf{x}_i, \mathbf{x}_{i-1}) \right\}$, where $C = C(\mathbf{x}_i, H_i)$ is the constant given by Lemma 3.1.

Suppose that $\mathbf{x}_1, \dots, \mathbf{x}_i$ are constructed for some $i \geq 1$. Then Lemma 3.1 provides a non-zero integer point $\mathbf{x}_{i+1}$ of arbitrarily large norm satisfying Conditions (a) to (c) as well as the left-hand side inequality of (d). If $i \geq 2$ the right-hand side inequality of (d) is also fulfilled for $\|\mathbf{x}_{i+1}\|$ large enough since $\lim_{X\to\infty} X\varphi(X) = \infty$.

The sequence $([\mathbf{x}_i])_{i\geq 1}$ converges in $\mathbb{P}^n(\mathbb{R})$ to a point ξ with

$$\operatorname{dist}(\xi, [\mathbf{x}_i]) \leq \sum_{j=i}^{\infty} \operatorname{dist}(\mathbf{x}_{j+1}, \mathbf{x}_j) \leq \operatorname{dist}(\mathbf{x}_{i+1}, \mathbf{x}_i) \sum_{j=0}^{\infty} 3^{-j} = \frac{3}{2}\operatorname{dist}(\mathbf{x}_{i+1}, \mathbf{x}_i) \tag{2}$$

for each $i \geq 1$. Moreover $\xi \in Z$ since Z is a closed subset of $\mathbb{P}^n(\mathbb{R})$ and $[\mathbf{x}_i] \in Z' \subseteq Z$ for each $i \geq 1$. When $i \geq 2$, Condition (d) combined with (2) yields

$$D_\xi(\mathbf{x}_i) = \|\mathbf{x}_i\| \operatorname{dist}(\xi, [\mathbf{x}_i]) \leq \frac{3}{2} \|\mathbf{x}_i\| \operatorname{dist}(\mathbf{x}_{i+1}, \mathbf{x}_i) \leq \varphi(\|\mathbf{x}_{i+1}\|). \tag{3}$$

In particular, we have $\lim_{i\to\infty} D_\xi(\mathbf{x}_i) = 0$. Let i_0, i be integers with $2 \leq i_0 \leq i$ such that $V := \mathbb{P}\Big(\langle\, \mathbf{x}_{i_0}, \dots, \mathbf{x}_i \rangle_{\mathbb{R}}\Big)$ is a strict subspace of $\mathbb{P}^n(\mathbb{R})$. By definition of H_i we have $V \subseteq H_i$. Moreover $[\mathbf{x}_i] \in H_i$, so that $s_H([\mathbf{x}_i]) \geq 1$, where $H = H_i$. Since $s_H \geq 0$, Condition (c) implies that there exists a smallest integer $\ell > i$ such that $H_\ell \neq H_i$, and therefore $[\mathbf{x}_\ell] \notin H_i$. In particular $[\mathbf{x}_\ell] \notin V$ and this proves that $(\mathbf{x}_i)_{i\geq i_0}$ spans $\mathbb{R}^{n+1}$ for each $i_0 \geq 2$. By [6, Lemma 6.2] we deduce that $\xi \in Z^{\mathrm{li}}$.

For each $X \geq \|\mathbf{x}_2\|$ we have $\|\mathbf{x}_i\| \leq X < \|\mathbf{x}_{i+1}\|$ for some $i \geq 2$ and using (3) we obtain

$$\mathcal{D}_\xi(X) \leq D_\xi(\mathbf{x}_i) \leq \varphi(\|\mathbf{x}_{i+1}\|) \leq \varphi(X).$$

Therefore the point $\xi \in Z^{\mathrm{li}}$ has the required property. By varying the sequence $(\mathbf{x}_i)_{i\geq 1}$, we obtain uncountably many such points as in the proof of [6, Proposition 10.4].

4. Proof of Theorem 1.1

Let k, n, N be positive integers with $N \geq 2$ and let $\varphi : (\mathbb{R}^n)^k \to \mathbb{R}^{N+1}$ be a k-linear map defined over $\mathbb{Q}$. We denote by $Z_{\mathbb{R}}$ (resp. $Z_{\mathbb{Q}}$) the projectivization of $\varphi((\mathbb{R}^n)^k)$ (resp. $\varphi((\mathbb{Q}^n)^k)$) and by S the set of points in $(\mathbb{Q}^n)^k$ whose image via φ is non-zero. For each $\alpha, \beta \in Z_{\mathbb{Q}}$, we define $m(\alpha, \beta)$ to be the largest integer $m \geq 0$ for which there exist $(\mathbf{x}_1, \dots, \mathbf{x}_k), (\mathbf{y}_1, \dots, \mathbf{y}_k) \in S$ satisfying

$$\alpha = [\varphi(\mathbf{x}_1, \dots, \mathbf{x}_k)], \quad \beta = [\varphi(\mathbf{y}_1, \dots, \mathbf{y}_k)], \quad \text{and} \quad \#\{i \in [1, k]\,;\, \mathbf{x}_i = \mathbf{y}_i\} = m. \tag{4}$$

Lemma 4.1. *Suppose that $\varphi((\mathbb{Q}^n)^k)$ spans the whole space $\mathbb{R}^{N+1}$. Let H be a projective proper subspace of $\mathbb{P}^N(\mathbb{R})$. For each $\alpha \in Z_{\mathbb{Q}}$ the set $Z_{\mathbb{Q}} \setminus H$ is non-empty and, upon defining*

$$s_H(\alpha) := \min\Big\{k - m(\alpha, \beta)\,;\, \beta \in Z_{\mathbb{Q}} \setminus H\Big\},$$

we have

(i) $s_H(\alpha) \geq 0$ *with equality if and only if* $\alpha \notin H$*;*

(ii) *if $s_H(\alpha) \geq 1$, then there exists a projective line $L \subseteq \mathbb{P}^N(\mathbb{R})$ defined over $\mathbb{Q}$ such that*

$$\alpha \in L(\mathbb{Q}) \subseteq Z_{\mathbb{Q}} \quad \textit{and} \quad \{y \in L(\mathbb{Q})\,;\, s_H(y) \geq s_H(\alpha)\} = \{\alpha\}. \tag{5}$$

By Lemma 4.1 the topological closure $Z = \overline{Z_{\mathbb{R}}}$ of $Z_{\mathbb{R}}$ satisfies the hypotheses of Theorem 2.1 with $Z' = Z_{\mathbb{Q}}$. This, in turn, implies Theorem 1.1.

Proof. Since by hypothesis $Z_{\mathbb{Q}}$ generates the whole space $\mathbb{P}^N(\mathbb{R})$ and since H is a strict subspace, there exist points $\beta \in Z_{\mathbb{Q}} \setminus H$. Assertion (i) is clear because $m(\alpha, \beta) = k$ if and only if $\alpha = \beta$.

To prove Assertion (ii), fix $\alpha \in Z_{\mathbb{Q}}$ with $s_H(\alpha) \geq 1$ and choose $\beta \in Z_{\mathbb{Q}} \setminus H$ such that

$$m := m(\alpha, \beta) = k - s_H(\alpha) \leq k - 1$$

is maximal. Choose $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ (resp. $(\mathbf{y}_1, \ldots, \mathbf{y}_k)$) in S satisfying (4) with α, β and m as above. Without lost of generality, we may assume that $\mathbf{x}_k \neq \mathbf{y}_k$. Then set

$$\begin{aligned} \boldsymbol{\alpha} &= \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_k), & \boldsymbol{\alpha}' &= \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{y}_k), \\ \boldsymbol{\beta} &= \varphi(\mathbf{y}_1, \ldots, \mathbf{y}_k), & \boldsymbol{\beta}' &= \varphi(\mathbf{y}_1, \ldots, \mathbf{y}_{k-1}, \mathbf{x}_k). \end{aligned}$$

By hypothesis, we have $\boldsymbol{\alpha}, \boldsymbol{\beta} \neq 0$, $\alpha = [\boldsymbol{\alpha}]$ and $\beta = [\boldsymbol{\beta}]$. For any $\lambda, \mu \in \mathbb{Q}$ the points

$$\lambda\boldsymbol{\alpha} + \mu\boldsymbol{\alpha}' = \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \lambda\mathbf{x}_k + \mu\mathbf{y}_k) \quad \text{and} \quad \mu\boldsymbol{\beta} + \lambda\boldsymbol{\beta}' = \varphi(\mathbf{y}_1, \ldots, \mathbf{y}_{k-1}, \lambda\mathbf{x}_k + \mu\mathbf{y}_k) \tag{6}$$

belong to $\varphi((\mathbb{Q}^n)^k)$. Suppose first that $\boldsymbol{\alpha}'$ is proportional to $\boldsymbol{\alpha}$. There exist infinitely many $\mu \in \mathbb{Q}^*$ such that $\boldsymbol{\alpha} + \mu\boldsymbol{\alpha}' \neq 0$ and $\mu\boldsymbol{\beta} + \boldsymbol{\beta}' \neq 0$. For those μ, the formulas (6) yield

$$m(\alpha, [\mu\boldsymbol{\beta} + \boldsymbol{\beta}']) = m([\boldsymbol{\alpha} + \mu\boldsymbol{\alpha}'], [\mu\boldsymbol{\beta} + \boldsymbol{\beta}']) \geq m + 1,$$

which implies that $[\mu\boldsymbol{\beta} + \boldsymbol{\beta}'] \in H$ by maximality of m. This is impossible since $[\boldsymbol{\beta}] = \beta \notin H$. Hence $\boldsymbol{\alpha}'$ is not proportional to $\boldsymbol{\alpha}$ and

$$L := \mathbb{P}(\langle\, \boldsymbol{\alpha}, \boldsymbol{\alpha}' \rangle_{\mathbb{R}})$$

is a projective line of $\mathbb{R}^N(\mathbb{R})$ defined over $\mathbb{Q}$, satisfying $\alpha \in L(\mathbb{Q}) \subseteq Z_{\mathbb{Q}}$. To conclude, it remains to show that L satisfies the second condition in (5). We first prove that for any $\lambda \in \mathbb{Q}$ we have

$$\boldsymbol{\beta} + \lambda\boldsymbol{\beta}' \neq 0 \quad \text{and} \quad [\boldsymbol{\beta} + \lambda\boldsymbol{\beta}'] \notin H. \tag{7}$$

It is true if $\boldsymbol{\beta}' = 0$. If $\boldsymbol{\beta}' \neq 0$, then $m(\alpha, [\boldsymbol{\beta}']) \geq m + 1$. This gives $[\boldsymbol{\beta}'] \in H$ by maximality of m, and (7) follows since $\beta = [\boldsymbol{\beta}] \notin H$. Combining (6) and (7), we get

$$m([\lambda\boldsymbol{\alpha} + \boldsymbol{\alpha}'], [\boldsymbol{\beta} + \lambda\boldsymbol{\beta}']) \geq m + 1,$$

which yields $s_H([\lambda\boldsymbol{\alpha} + \boldsymbol{\alpha}']) \leq k - m - 1 = s_H(\alpha) - 1$. □

5. Two examples

In this section we give two examples of sets to which Theorem 1.1 applies.

The sets $G_{k,n}$. Let k, n be two integers with $1 \leq k < n$ and $n \geq 3$. We define the Grassmannian $G_{k,n}$ as the projectivization of the set

$$\{\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \mid \mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n\}$$

inside $\mathbb{P}(\bigwedge^k \mathbb{R}^n)$. By identifying $\bigwedge^k \mathbb{R}^n$ to $\mathbb{R}^N$ via an ordering of the Plücker coordinates, where $N = \binom{n}{k} \geq 3$, the set $G_{k,n}$ is identified to a subset $\mathbb{P}(\mathbb{R}^N)$. It is Zariski closed, thus closed. By construction it is the projectivization of the image of the k-linear map $\varphi : (\mathbb{R}^n)^k \to \mathbb{R}^N$ defined by

$$\varphi(\mathbf{x}_1, \dots, \mathbf{x}_k) = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k, \quad \mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n.$$

It is easily seen that $\varphi((\mathbb{Q})^n)$ spans $\mathbb{R}^N$, so that the conditions of Lemma 4.1 are fulfilled and we get $\widehat{\lambda}(G_{k,n}) = 1$.

The sets $H_{n,k}$. Let k, n be positive integers with $n \geq 2$ and $n + k \geq 4$. The vector space $\mathbb{R}[x_1, \dots, x_n]_k$ of homogenous polynomials of degree k in n variables admits for basis the set of monomials

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \alpha_1 + \cdots + \alpha_n = k$$

which identifies it to $\mathbb{R}^N$, where $N = \binom{n+k-1}{k} \geq 3$. We denote by $H_{n,k} \subseteq \mathbb{P}^N(\mathbb{R})$ the projectivization of the set

$$\Big\{ \prod_{j=1}^{k} L_j \mid L_1, \dots, L_k \in \mathbb{R}[x_1, \dots, x_n]_1 \Big\} = \psi((\mathbb{R}[x_1, \dots, x_n]_1)^k),$$

where the k-linear map $\psi : (\mathbb{R}[x_1, \dots, x_n]_1)^k \to \mathbb{R}[x_1, \dots, x_n]_k \cong \mathbb{R}^N$ is defined by

$$\psi(L_1, \dots, L_k) = L_1 \cdots L_k$$

for any $L_1, \dots, L_k \in \mathbb{R}[x_1, \dots, x_n]_1$. It can be shown that $H_{n,k}$ is closed. Since $\psi((\mathbb{Q}[x_1, \dots, x_n]_1)^k)$ spans $\mathbb{R}^N$, we can once again apply Lemma 4.1 to get $\widehat{\lambda}(H_{n,k}) = 1$.

6. Quadratic hypersurfaces of Witt index > 1

Let $n \geq 1$ be an integer. A quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ is a non-empty subset which is the set of zeros in $\mathbb{P}^n(\mathbb{R})$ of an irreductible homogeneous polynomial q of $\mathbb{Q}[t_0, \dots, t_n]$ of degree 2. The Witt index (over $\mathbb{Q}$) m of q is the largest integer $m \geq 0$ such that $\mathbb{Q}^{n+1}$ contains an orthogonal sum (with respect to the symmetric bilinear form associated to q) of m hyperbolic planes for q.

The following result is part of [6, Theorem 1.1].

Theorem 6.1 (Poëls-Roy, 2019). *Let $n \geq 3$ be an integer and let Z be a quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$, and let m be the Witt index (over $\mathbb{Q}$) of the quadratic form on $\mathbb{Q}^{n+1}$ defining Z. If $m \geq 2$, then there are uncountably many $\xi \in Z^{\mathrm{li}}$ such that $\widehat{\lambda}(\xi) = 1$, and thus $\widehat{\lambda}(Z) = 1$.*

In this section we use our Theorem 2.1 to give an alternative shorter proof of this statement. We choose $Z' = Z(\mathbb{Q})$ and for each proper projective subspace $H \subseteq \mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ and each $\alpha \in Z(\mathbb{Q})$ we set

$$s_H(\alpha) = \begin{cases} 0 \text{ if } \alpha \notin H, \\ 1 \text{ if } \alpha \in H \text{ and } \langle\alpha\rangle^{\perp} \neq H, \\ 2 \text{ if } \alpha \in H \text{ and } \langle\alpha\rangle^{\perp} = H, \end{cases} \tag{8}$$

where for any $\beta \in \mathbb{P}^n(\mathbb{R})$ the set $\langle\beta\rangle^{\perp}$ denotes the orthogonal in $\mathbb{P}^n(\mathbb{R})$ of β (with respect to the symmetric bilinear form associated to q). This choice of s_H satisfies Condition (i) of Theorem 2.1. If $s_H(\alpha) \geq 1$ we show that there exists a projective line L defined over $\mathbb{Q}$ with the following properties stronger than Condition (ii) of Theorem 2.1:

(I) $\alpha \in L(\mathbb{Q}) \subseteq Z(\mathbb{Q})$;
(II) $s_H(\beta) < s_H(\alpha)$ for any $\beta \in L(\mathbb{Q}) \setminus \{\alpha\}$.

If $s_H(\alpha) = 2$, we choose a projective line L defined over $\mathbb{Q}$ satisfying (I) (this exists since the Witt index m of q is ≥ 2). We claim that L also satisfies (II). Indeed, since $H^{\perp} = \{\alpha\}$, we have $\langle\beta\rangle^{\perp} \neq \langle\alpha\rangle^{\perp} = H$ for any $\beta \neq \alpha$, and so (II) follows.

If $s_H(\alpha) = 1$, the situation is more complicated. Arguing as in the proof of [6, Lemma 10.3], we note that there exists a zero β of q in $Z(\mathbb{Q}) \cap \langle\alpha\rangle^{\perp} \setminus H$. Since $\alpha \in H$ and $\beta \notin H$, any point $\gamma \neq \alpha$ in the projective line $L \subseteq Z$ generated by α and β does not belong to H and thus satisfies $s_H(\gamma) = 0 < s_H(\alpha)$.

Remark. In the present paper and in [6], we search for points of a projective (not necessarily non-singular) quadratic hypersurface Z which are *very well approximated* by rational points of the ambient space. More precisely we look for points whose uniform exponent of rational simultaneous approximation is maximal. When the Witt index m of Z is ≤ 1, we show in [6] that the best rational approximations to such points are necessarily outside Z. On the contrary, when $m > 1$ as in Theorem 6.1, the best rational approximations are necessarily points of Z. The authors of [4] consider the problem of simultaneous approximation to real points on a projective non-singular quadratic hypersurface Z by rational points of Z, they call that *intrinsic approximation*. Moreover, their goal is in a sense opposite to ours; they show that all points of Z admit intrinsic rational approximations to a certain precision, and compute the Hausdorff dimension of the set of *badly approximable* points. It is interesting to note that in [4], the rational Witt index (called $\mathbb{Q}$–rank) also plays an important role.

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