A new exponent of simultaneous rational approximation

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Abstract

We introduce a new exponent of simultaneous rational approximation $\widehat{\lambda}_{\min}(\xi,\eta)$ for pairs of real numbers ξ,η, in complement to the classical exponents $\lambda(\xi,\eta)$ of best approximation, and $\widehat{\lambda}(\xi,\eta)$ of uniform approximation. It generalizes Fischler's exponent $\beta_0(\xi)$ in the sense that $\widehat{\lambda}_{\min}(\xi,\xi^2) = 1/\beta_0(\xi)$ whenever $\lambda(\xi,\xi^2) = 1$. Using parametric geometry of numbers, we provide a complete description of the set of values taken by $(\lambda,\widehat{\lambda}_{\min})$ at pairs (ξ,η) with $1,\xi,\eta$ linearly independent over $\mathbb{Q}$.

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1 Introduction

Let ξ and η be non-zero real numbers. The following simultaneous approximation problem has been intensively studied during the last decades:

Problem $E_{\lambda,X}$: Given $\lambda > 0$ and $X \geq 1$, we search for solutions $(x_0,x_1,x_2) \in \mathbb{Z}^3 \setminus \{0\}$ of the system

$$1 \leq |x_0| \leq X \quad \text{and} \quad \max(|x_0\xi - x_1|, |x_0\eta - x_2|) \leq X^{-\lambda}.$$

We denote by $\lambda(\xi,\eta)$ (resp. $\widehat{\lambda}(\xi,\eta)$) the supremum of real numbers λ for which $E_{\lambda,X}$ admits a non-zero integer solution for arbitrarily large values of X (resp. for each sufficiently large value of X). For all real numbers ξ,η, we have

$$\lambda(\xi,\eta) \geq \widehat{\lambda}(\xi,\eta) \geq \frac{1}{2},$$

the right-hand side inequality following from Dirichlet's box principle (or, equivalently, Minkowski's theorem). The study of such Diophantine exponents of approximation goes back to Jarník and Khinchin, see [1] for a well supplied account of the topic. In this paper, we consider the following variant:

Problem $E_{\lambda,\mu,X}$: Given $\lambda > 0$, $\mu \geq 0$ and $X > 1$, we search for solutions $(x_0,x_1,x_2) \in \mathbb{Z}^3 \setminus \{0\}$ of the system

$$1 \leq |x_0| \leq X \quad \text{and} \quad \max(|x_0\xi - x_1|, |x_0\eta - x_2|) \leq \min(X^{-\lambda}, |x_0|^{-\mu}).$$

This was introduced by Fischler in [7] in the special case where $\eta = \xi^2$. For $0 \leq \mu < \lambda(\xi,\eta)$, we denote by $\widehat{\lambda}_\mu(\xi,\eta)$ the supremum of the real numbers λ for which $E_{\lambda,\mu,X}$ admits a non-zero integer solution for each sufficiently large value of X. Note that the map $\mu \mapsto \widehat{\lambda}_\mu(\xi,\eta)$ is non-increasing. We define

$$\widehat{\lambda}_{\min}(\xi,\eta) = \inf_{0<\mu<\lambda(\xi,\eta)} \widehat{\lambda}_\mu(\xi,\eta) = \lim_{\mu\to\lambda(\xi,\eta)^-} \widehat{\lambda}_\mu(\xi,\eta). \tag{1.1}$$

See Remark 2.2 and (2.2) for an interpretation of $\widehat{\lambda}_{\min}$. Note that for $\mu = 0$ we have $\widehat{\lambda}_0(\xi,\eta) = \widehat{\lambda}(\xi,\eta)$, so that

$$\widehat{\lambda}_{\min}(\xi,\eta) \le \widehat{\lambda}(\xi,\eta).$$

More generally, we have $\widehat{\lambda}_\mu(\xi,\eta) = \widehat{\lambda}(\xi,\eta)$ for any $\mu < \widehat{\lambda}(\xi,\eta)$. In particular, if $\widehat{\lambda}(\xi,\eta) = \lambda(\xi,\eta)$, Definition (1.1) gives $\widehat{\lambda}_{\min}(\xi,\eta) = \widehat{\lambda}(\xi,\eta) = \lambda(\xi,\eta)$. Yet, it is well-known that $\widehat{\lambda}(\xi,\eta) = \lambda(\xi,\eta) = 1/2$ for almost all (ξ,η) with respect to the Lebesgue measure on $\mathbb{R}^2$ (see [3, §2]). We thus have the following result:

Theorem 1.1. *For almost all real numbers ξ,η (with respect to the Lebesgue measure on $\mathbb{R}^2$), we have*

$$\widehat{\lambda}_{\min}(\xi,\eta) = \frac{1}{2}.$$

The goal of this paper is to give an interpretation of the exponents $\widehat{\lambda}_\mu(\xi,\eta)$ and $\widehat{\lambda}_{\min}(\xi,\eta)$ in the setting of parametric geometry of numbers and to prove the following description for the spectrum of the pair $(\lambda,\widehat{\lambda}_{\min})$, *i.e.* the set of values taken by $(\lambda,\widehat{\lambda}_{\min})$ at pairs (ξ,η) with $1,\xi,\eta$ linearly independent over $\mathbb{Q}$.

Theorem 1.2. *For any $\xi,\eta \in \mathbb{R}$ with $1,\xi,\eta$ linearly independent over $\mathbb{Q}$, we have either $\widehat{\lambda}_{\min}(\xi,\eta) = \lambda(\xi,\eta) = 1/2$, or*

$$0 \le \widehat{\lambda}_{\min}(\xi,\eta) \le 1, \quad \frac{1}{2} < \lambda(\xi,\eta) \le +\infty \quad \text{and} \quad \frac{\widehat{\lambda}_{\min}(\xi,\eta)^2}{1-\widehat{\lambda}_{\min}(\xi,\eta)} \le \lambda(\xi,\eta). \tag{1.2}$$

Conversely, for any $\widehat{\lambda} \in \mathbb{R}$ and any $\lambda \in \mathbb{R}\cup\{+\infty\}$ satisfying either $\widehat{\lambda} = \lambda = 1/2$, or

$$0 \le \widehat{\lambda} \le 1, \quad \frac{1}{2} < \lambda \le +\infty \quad \text{and} \quad \frac{\widehat{\lambda}^2}{1-\widehat{\lambda}} \le \lambda, \tag{1.3}$$

there exist two real numbers ξ and η, with $1,\xi,\eta$ linearly independent over $\mathbb{Q}$, such that

$$\lambda(\xi,\eta) = \lambda \quad \text{and} \quad \widehat{\lambda}_{\min}(\xi,\eta) = \widehat{\lambda}.$$

Laurent computed the spectrum of $(\lambda,\widehat{\lambda})$ in [8] (see Corollary 2 of [8]). He proved that for any ξ,η with $1,\xi,\eta$ linearly independent over $\mathbb{Q}$, we have

$$\frac{1}{2} \le \widehat{\lambda}(\xi,\eta) \le 1, \quad \frac{\widehat{\lambda}(\xi,\eta)^2}{1-\widehat{\lambda}(\xi,\eta)} \le \lambda(\xi,\eta) \le +\infty, \tag{1.4}$$

and that (1.4) describe entirely the spectrum of $(\lambda,\widehat{\lambda})$. Since $\widehat{\lambda}_{\min} \le \widehat{\lambda}$, the inequalities (1.2) are implied by (1.4) together with $\lambda(\xi,\eta) > 1/2$. It would be interesting to study the joint spectrum of $(\lambda,\widehat{\lambda},\widehat{\lambda}_{\min})$.
In [7] Fischler introduced a new exponent of approximation $\beta_0(\xi)$ for each real number ξ. When $\lambda(\xi,\xi^2) < 1$, he defined $\beta_0(\xi) = +\infty$. Otherwise he set $\beta_0(\xi) = \lim_{\varepsilon\to 0^+}\beta_\varepsilon(\xi)$, with $\beta_\varepsilon(\xi) = 1/\widehat{\lambda}_{1-\varepsilon}(\xi,\xi^2)$ (for $0 < \varepsilon \le 1$). Then he studied in depth the real numbers ξ for which $\beta_0(\xi) < 2$. For those numbers, the exponent $\beta_0(\xi)$ and $\widehat{\lambda}_{\min}(\xi,\xi^2)$ are related as follows.

Lemma 1.1. *If $\beta_0(\xi) < 2$, then $\lambda(\xi,\xi^2) = 1$ and $\beta_0(\xi) = 1/\widehat{\lambda}_{\min}(\xi,\xi^2)$.*

Proof Let ξ be such that $\beta_0(\xi) < 2$. Then we have $\lambda(\xi,\xi^2) \geq 1$. In general, the inequality $1/\beta_0(\xi) \leq \widehat{\lambda}(\xi,\xi^2)$ holds, so that $\widehat{\lambda}(\xi,\xi^2) > 1/2$. This implies that $\lambda(\xi,\xi^2) \leq 1$. This result can be obtained from Davenport and Schmidt's work by generalizing Lemmas 2 and 6 of [6] (see for example [10, Corollaire 6.2.7]); it is also a corollary of a recent result due to Schleischitz [13, Theorem 1.6]. Finally, this shows that $\lambda(\xi,\xi^2) = 1$. In this case, we have $\beta_0(\xi) = 1/\widehat{\lambda}_{\min}(\xi,\xi^2)$ by definition of $\beta_\varepsilon(\xi)$ $(0 < \varepsilon < 1)$.

□

Let $\mathcal{V}$ denotes the set

$$\mathcal{V} = \big\{(\xi,\eta) \mid 1,\xi,\eta \text{ linearly independent over } \mathbb{Q} \text{ and } \lambda(\xi,\eta) = 1\big\}. \quad (1.5)$$

Applying Theorem 1.2 with $\lambda = 1$, we obtain the following result.

Theorem 1.3. *With the above notation, the set of values taken by $1/\widehat{\lambda}_{\min}$ at pairs $(\xi,\eta) \in \mathcal{V}$ is $[\gamma,+\infty]$, where $\gamma = (1+\sqrt{5})/2$ denotes the golden ratio.*

The situation is radically different for the pairs (ξ,ξ^2). Following [2] let us denote by $\mathcal{S}$ the set of all values $\sigma = 1/\limsup_{k\to+\infty}[s_{k+1}; s_k,\dots,s_1]$ where $(s_k)_{k\geq 1}$ runs through all sequences of positive integers (here $[a_0; a_1, a_2, \dots]$ denotes the continued fraction whose partial quotients are $a_0, a_1, \dots$). The largest element of $\mathcal{S}$ is $\frac{1}{\gamma}$. The values immediately below have been described by Cassaigne [4]. They constitute a decreasing sequence of quadratic numbers converging to the largest accumulation point $s \approx 0.3867\dots$ of $\mathcal{S}$. Also note that Cassaigne has shown in [4] that this set has empty interior. Elements of $\mathcal{S}$ appear in the description of the classical exponents of approximation to Sturmian continued fractions, studied by Bugeaud and Laurent in [2], and of Sturmian type numbers (see [9]). The set $\mathcal{S}$ is related to the spectrum of β_0 by the following result (see [7]):

Theorem 1.4 (Fischler, 2007)**.** *Let us set $\mathcal{S}_0 = \{\beta_0(\xi) \mid \xi \in \mathbb{R}$ not algebraic of degree $\leq 2\}$. Then we have*

$$\mathcal{S}_0 \cap [\gamma,\sqrt{3}) = \Big\{1 + \frac{1}{1+\sigma} \mid \sigma \in \mathcal{S}\Big\} \cap (1,\sqrt{3}).$$

In view of the description of $\mathcal{S}$ given above, the smallest element of $\mathcal{S}_0$ in $[\gamma,+\infty)$ is therefore γ and the values immediately above constitute an increasing sequence of quadratic numbers converging to the smallest accumulation point $1.721\dots < \sqrt{3}$ of $\mathcal{S}_0$. Thus, Theorem 1.3 implies that $\{1/\widehat{\lambda}_{\min}(\xi,\eta) \mid (\xi,\eta) \in \mathcal{V}\} \cap [\gamma,\sqrt{3}]$ is the full interval $[\gamma,\sqrt{3}]$ (where $\mathcal{V}$ is defined by (1.5)), whereas

$$\{\beta_0(\xi) \mid \xi \in \mathbb{R} \text{ not algebraic of degree } \leq 2\} \cap [\gamma,\sqrt{3}] = \{1/\widehat{\lambda}_{\min}(\xi,\xi^2) \mid (\xi,\xi^2) \in \mathcal{V}\} \cap [\gamma,\sqrt{3}]$$

has empty interior and its complement in $[\gamma,\sqrt{3}]$ has non-empty interior by Theorem 1.4.

In Section 2 we study the "rigidity" of the exponents $\widehat{\lambda}_\mu$ and we recall the notion of *minimal points*, which is useful to compute the exponents. We use parametric geometry of numbers to prove Theorem 1.2. In section 3 we briefly recall the elements of that theory and we provide a parametric version of the exponent $\widehat{\lambda}_{\min}(\xi,\eta)$. The proof of Theorem 1.2 is given in Section 4.

2 Exponents $\widehat{\lambda}_\mu$

Unlike the classical exponents $\lambda(\xi,\eta)$ and $\widehat{\lambda}(\xi,\eta)$, the exponent $\widehat{\lambda}_\mu(\xi,\eta)$ may change if we perturbate the problem $E_{\lambda,\mu,X}$ slightly, for example by using $\|\mathbf{x}\|$ instead of $|x_0|$ (where $\|\cdot\|$ is a fixed norm on $\mathbb{R}^3$). However, as kindly pointed out to the author by Damien Roy, this happens only at the points μ at which the non-increasing map $\mu \mapsto \widehat{\lambda}_\mu(\xi,\eta)$ is not continuous; this set of points is therefore countable. We formalize this claim in Proposition 2.1 below. Thus the exponent

$\widehat{\lambda}_{\min}(\xi,\eta)$ can be defined using any norm $\|\mathbf{x}\|$ instead of $|x_0|$ in $E_{\lambda,\mu,X}$.

Let $\xi,\eta\in\mathbb{R}$ be two real numbers with $1,\xi,\eta$ linearly independent over $\mathbb{Q}$. The exponents $\widehat{\lambda}_\mu(\xi,\eta)$ are defined as in the introduction. We denote by $\|\cdot\|$ the usual Euclidean norm in $\mathbb{R}^3$. If $f,g: I\to[0,+\infty)$ are two fonctions on a set I, we write $f\ll g$ (resp. $f\gg g$) to mean that there is a positive constant c such that $f(x)\leq cg(x)$ (resp. $f(x)\geq cg(x)$) for each $x\in I$. We write $f\asymp g$ if both $f\ll g$ and $g\ll f$ hold. Let $\Delta,\mathrm{N}:\mathbb{R}^3\to[0,+\infty)$ such that for any $\mathbf{x}=(x_0,x_1,x_2)$

$$\Delta(\mathbf{x})\asymp\max\big(|x_0\xi-x_1|,|x_0\eta-x_2|\big)$$

and

$$\mathrm{N}(\mathbf{x})\asymp\|\mathbf{x}\|\quad\text{if }\max\big(|x_0\xi-x_1|,|x_0\eta-x_2|\big)<1,$$

(the implicit constants depend only on Δ,N,ξ and η). Note that we may take $\mathrm{N}(x)=|x_0|$, although N is not a norm in this case. For $0\leq\mu<\lambda(\xi,\eta)$, we denote by $\widehat{\nu}_\mu(\xi,\eta)$ the supremum of the real numbers ν for which the system

$$\mathrm{N}(\mathbf{x})\leq X\quad\text{and}\quad\Delta(\mathbf{x})\leq\min\big(X^{-\nu},\mathrm{N}(\mathbf{x})^{-\mu}\big)\tag{2.1}$$

admits a non-zero integer solution for each sufficiently large value of X. If there is no such real number ν, we set $\widehat{\nu}_\mu(\xi,\eta)=0$. The map $\mu\mapsto\widehat{\nu}_\mu(\xi,\eta)$ is non-increasing. We set

$$\widehat{\nu}_{\min}(\xi,\eta)=\inf_{0<\mu<\lambda(\xi,\eta)}\widehat{\nu}_\mu(\xi,\eta)=\lim_{\mu\to\lambda(\xi,\eta)^-}\widehat{\nu}_\mu(\xi,\eta).$$

We have the following result:

Proposition 2.1. *The non-increasing maps $\mu\mapsto\widehat{\nu}_\mu(\xi,\eta)$ and $\mu\mapsto\widehat{\lambda}_\mu(\xi,\eta)$ have the same set of discontinuities on $[0,+\infty)$ and they coincide outside of this set. Moreover we have:*

$$\widehat{\nu}_{\min}(\xi,\eta)=\widehat{\lambda}_{\min}(\xi,\eta).$$

Proof Let us prove that $\widehat{\lambda}_{\mu'}(\xi,\eta)\geq\widehat{\nu}_\mu(\xi,\eta)$ for any $0\leq\mu'<\mu$. If $\widehat{\nu}_\mu(\xi,\eta)=0$ it is trivial. Now, suppose $\widehat{\nu}_\mu(\xi,\eta)>0$ and let $0<\lambda'<\lambda<\widehat{\nu}_\mu(\xi,\eta)$. If X is large enough, then (2.1) has a non-zero integer solution $\mathbf{x}$ and this point $\mathbf{x}$ is also solution of the problem $E_{\lambda',\mu',X}$ stated in the introduction. By letting λ' tend to λ, then by letting λ tend to $\widehat{\nu}_\mu(\xi,\eta)$, it follows that $\widehat{\lambda}_{\mu'}(\xi,\eta)\geq\widehat{\nu}_\mu(\xi,\eta)$. Conversely, we also have $\widehat{\nu}_{\mu'}(\xi,\eta)\geq\widehat{\lambda}_\mu(\xi,\eta)$. In summary, we have shown that for any $\mu_1<\mu<\mu_2$, we have $\widehat{\lambda}_{\mu_2}(\xi,\eta)\leq\widehat{\nu}_\mu(\xi,\eta)\leq\widehat{\lambda}_{\mu_1}(\xi,\eta)$, which yields $\widehat{\nu}_\mu(\xi,\eta)=\widehat{\lambda}_\mu(\xi,\eta)$ at each point where $\mu\mapsto\widehat{\lambda}_\mu(\xi,\eta)$ (or $\mu\mapsto\widehat{\nu}_\mu(\xi,\eta)$) is continuous. □

To compute the exponent $\widehat{\nu}_\mu(\xi,\eta)$ it is sufficient to consider only "the best" solutions of (2.1). Following Davenport and Schmidt [5], [6], we call a sequence of *minimal points* (associated to N and Δ) a sequence of non-zero integer points $(\mathbf{x}_i)_{i\geq0}$ which satisfies

- $\mathrm{N}(\mathbf{x}_1)<\mathrm{N}(\mathbf{x}_2)<\dots$ and $\Delta(\mathbf{x}_1)>\Delta(\mathbf{x}_2)>\dots,$
- For each non-zero $\mathbf{z}\in\mathbb{Z}^3$, if $\mathrm{N}(\mathbf{z})<\mathrm{N}(\mathbf{x}_{i+1})$, then $\Delta(\mathbf{z})\geq\Delta(\mathbf{x}_i)$.

For simplicity, let us write $\mathrm{X}_i=\mathrm{N}(\mathbf{x}_i)$ and $\Delta_i=\Delta(\mathbf{x}_i)$. Let $\mu\geq0$, $\lambda>0$ and $X>0$, and suppose that $\mathbf{x}\in\mathbb{Z}^3$ satisfies (2.1). If $\mathrm{N}(\mathbf{x})\gg1$, then there is an index i such that $\mathrm{X}_i\leq\mathrm{N}(\mathbf{x})<\mathrm{X}_{i+1}$. Since $\Delta_i\leq\Delta(\mathbf{x})$ and $\mu\geq0$, the point $\mathbf{x}_i$ is also solution of (2.1). Hence, for each $\mu<\lambda(\xi,\eta)$, the exponent $\widehat{\nu}_\mu(\xi,\eta)$ is the supremum of the real numbers λ such that for each X large enough, there exists $i\geq1$ for which

$$\mathrm{X}_i\leq X\quad\text{and}\quad\Delta_i\leq\min\big(X^{-\lambda},\mathrm{X}_i^{-\mu}\big).$$

Let $0 < i_1 < i_2 < \dots$ denote the sequence of indices i such that $\Delta_i \leq \mathrm{X}_i^{-\mu}$. Then

$$\widehat{\nu}_\mu(\xi,\eta) = \liminf_{k\to\infty} \frac{-\log(\Delta_{i_k})}{\log(\mathrm{X}_{i_{k+1}})}. \tag{2.2}$$

Remark 2.2. This formula is similar to (11) of [7]. Roughly speaking, $\widehat{\lambda}_{\min}(\xi,\eta)$ corresponds to $\widehat{\lambda}(\xi,\eta)$ when we only take in account the exceptionally precise approximants, *i.e.* solutions $\mathbf{x} = (x_0,x_1,x_2) \in \mathbb{Z}^3$ of $E_{\lambda,X}$ with $\max(|x_0\xi - x_1|, |x_0\eta - x_2|)$ very close to $|x_0|^{-\lambda(\xi,\eta)}$.

3 Parametric geometry of numbers

3.1 The setting

In this section we quickly recall the basics of the parametric geometry of numbers following Schmidt and Summerer [14], [15] and Roy [11]. We use the setting of [11]. We denote by $\mathbf{x}\wedge\mathbf{y}$ the standard vector product of two vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}^3$, by $\mathbf{x}\cdot\mathbf{y}$ their standard inner product and by $\|\mathbf{x}\|$ the Euclidean norm of $\mathbf{x}$. Fix $\mathbf{u} \in \mathbb{R}^3 \setminus \{0\}$. For each $q \geq 0$ we set

$$\mathcal{C}_\mathbf{u}(q) := \{\mathbf{x} \in \mathbb{R}^3 \,;\, \|\mathbf{x}\| \leq 1, |\mathbf{x}\cdot\mathbf{u}| \leq e^{-q}\} \quad\text{and}\quad \mathcal{C}^*_\mathbf{u}(q) := \{\mathbf{x} \in \mathbb{R}^3 \,;\, \|\mathbf{x}\| \leq e^q, \|\mathbf{x}\wedge\mathbf{u}\| \leq 1\}.$$

For $j = 1,2,3$ we define a function $L_j : [0,+\infty) \to \mathbb{R}$ by $L_j(q) = \log(\lambda_{j,\mathbf{u}}(q))$, where $\lambda_{j,\mathbf{u}}(q)$ denotes the j-th successive minimum of the convex body $\mathcal{C}_\mathbf{u}(q)$ with respect to the lattice $\mathbb{Z}^3$. We set $\mathbf{L}_\mathbf{u} = (L_1,L_2,L_3)$. The functions L_j are continuous, piecewise linear with slopes 0 and 1, and by Minkowski's second theorem they satisfy $L_1(q) + L_2(q) + L_3(q) = q + \mathcal{O}(1)$ (for any $q \geq 0$). For each $\mathbf{x} \in \mathbb{R}^3$, we further define $\lambda_\mathbf{x}(q, \mathcal{C}_\mathbf{u}(q))$ to be the smallest real number $\lambda \geq 0$ such that $\mathbf{x} \in \lambda\mathcal{C}_\mathbf{u}(q)$. When $\mathbf{x} \neq 0$, this number is positive and so we obtain a function $L_\mathbf{x} : [0,+\infty) \to \mathbb{R}$ by putting $L_\mathbf{x}(q) := \log(\lambda_\mathbf{x}(q, \mathcal{C}_\mathbf{u}(q)))$. For $j = 1,2,3$ we set

$$\overline{\psi}_j(\mathbf{u}) = \overline{\psi}_j = \limsup_{q\to\infty} \frac{L_j(q)}{q} \quad\text{and}\quad \underline{\psi}_j(\mathbf{u}) = \underline{\psi}_j = \liminf_{q\to\infty} \frac{L_j(q)}{q}.$$

Similarly we define the function $\mathbf{L}^*_\mathbf{u} = (L^*_1, L^*_2, L^*_3)$, $\overline{\psi}^*_j$, $L^*_\mathbf{x}$ ($\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$), $\underline{\psi}^*_j$ associated to the family of convex bodies $\mathcal{C}^*_\mathbf{u}(q)$. For any non-zero $\mathbf{x} \in \mathbb{R}^3$ we have

$$L^*_\mathbf{x}(q) = \max\big(\log\|\mathbf{x}\wedge\mathbf{u}\|, \log\|\mathbf{x}\| - q\big) \quad (q \geq 0). \tag{3.1}$$

The dual functions L^*_j are related to the functions L_j by Mahler's duality:

Proposition 3.1 (Mahler)**.** *For $j = 1,2,3$ we have $\mathrm{L}_j(q) = -\mathrm{L}^*_{4-j}(q) + \mathcal{O}(1)$ for all $q > 0$.*

Thus

$$\underline{\psi}_j = -\overline{\psi}^*_{4-j} \quad\text{and}\quad \overline{\psi}_j = -\underline{\psi}^*_{4-j} \quad (j = 1,2,3). \tag{3.2}$$

The following definition is that of a 3-system (see [12, Definition 4.1]; this is an analog of a $(3,0)$-system for Schmidt and Summerer [15]).

Definition 3.2. Fix a real number $q_0 \geq 0$. A 3-system on $[q_0,+\infty)$ is a continuous piecewise linear map $\mathbf{P} = (P_1,P_2,P_3) : [q_0,+\infty) \to \mathbb{R}^3$ with the following properties:

(a) For each $q \geq q_0$, we have $0 \leq P_1(q) \leq P_2(q) \leq P_3(q)$ and $P_1(q) + P_2(q) + P_3(q) = q$.

(b) If H is a non-empty open subinterval of $[q_0,+\infty)$ on which $\mathbf{P}$ is differentiable, then there is an integer r ($1 \leq r \leq 3$), such that P_r has slope 1 on H while the other components P_j of $\mathbf{P}$ ($j \neq r$) are constant on H.

(c) If $q > q_0$ is a point at which $\mathbf{P}$ is not differentiable and if the integers r and s, for which P_r has slope 1 on $(q-\varepsilon, q)$ and P_s has slope 1 on $(q, q+\varepsilon)$ (for $\varepsilon > 0$ small enough), satisfy $r < s$, then we have $P_r(q) = P_{r+1}(q) = \cdots = P_s(q)$.

The following fondamental result was proved by Roy in [11].

Theorem 3.1 (Roy, 2015)**.** *For each non-zero point $\mathbf{u} \in \mathbb{R}^3$, there exist $q_0 > 0$ and a 3-system $\mathbf{P}$ on $[q_0, +\infty)$ such that $\|\mathbf{L_u} - \mathbf{P}\|_\infty$ is bounded on $[q_0, +\infty)$. Conversely, for each 3-system $\mathbf{P}$ on an interval $[q_0, +\infty)$, there exists a non-zero point $\mathbf{u} \in \mathbb{R}^3$ such that $\|\mathbf{L_u} - \mathbf{P}\|_\infty$ is bounded on $[q_0, +\infty)$.*

Following [15, §3] we define the *combined graph* of a set of real valued functions defined on an interval I to be the union of their graphs in $I \times \mathbb{R}$. For a map $\mathbf{P} : [c, +\infty) \to \mathbb{R}^3$ and an interval $I \subset [c, +\infty)$, we also define the *combined graph of* $\mathbf{P}$ *on* I to be the combined graph of its components P_1, P_2, P_3 restricted to I.

We recall the following relationship between classical and parametric exponents (see [11]). For any $\mathbf{u} = (1, \xi, \eta)$ with $\mathbb{Q}$-linearly independent coordinates, we have

$$\big(\underline{\psi}_3(\mathbf{u}), \overline{\psi}_3(\mathbf{u})\big) = \Big(\frac{\widehat{\lambda}(\xi,\eta)}{1+\widehat{\lambda}(\xi,\eta)}, \frac{\lambda(\xi,\eta)}{1+\lambda(\xi,\eta)}\Big). \tag{3.3}$$

3.2 Parametric formulation of $\widehat{\lambda}_{\min}$

Definition 3.3. Let $c \geq 0$ and let $P : [c, +\infty) \to [0, +\infty)$ be an unbounded continuous piecewise linear function, with slopes 0 and 1. Let $(q_i)_{i\geq 0}$ be the increasing sequence of abscissas at which P changes slope from 1 to 0. We suppose $(q_i)_{i\geq 0}$ infinite and define $\overline{\psi}(P)$, $\underline{\psi}(P)$ by

$$\overline{\psi}(P) = \limsup_{q\to+\infty} \frac{P(q)}{q} = \limsup_{k\to\infty} \frac{P(q_k)}{q_k} \quad \text{and} \quad \underline{\psi}(P) = \liminf_{q\to+\infty} \frac{P(q)}{q}.$$

For each $\alpha < \overline{\psi}(P)$, let $(q_{i,\alpha})_{i\geq 0}$ be the (increasing) subsequence of all abscissas q_k satisfying $q_k^{-1}P(q_k) \geq \alpha$. For each $i \geq 0$ we denote by $r_{i,\alpha}$ the abscissa of the intersection point of the horizontal line passing through $(q_{i,\alpha}, P(q_{i,\alpha}))$ and of the line with slope 1 passing through $(q_{i+1,\alpha}, P(q_{i+1,\alpha}))$. We set

$$\kappa_\alpha(P) = \liminf_{i\to+\infty} \frac{P(q_{i,\alpha})}{r_{i,\alpha}} \quad \text{and} \quad \kappa(P) = \lim_{\alpha\to\overline{\psi}(P)} \kappa_\alpha(P).$$

Let $P^* : [c, +\infty) \to (-\infty, 0]$ be an unbounded continuous piecewise linear function, with slopes 0 and -1 and which changes from slope -1 to 0 infinitely many times. In a dual manner, for $\alpha > \liminf_{q\to\infty} P^*(q)/q$ we define

$$\kappa^*_\alpha(P^*) = -\kappa_{-\alpha}(-P^*) \quad \text{and} \quad \kappa^*(P^*) = -\kappa(-P^*).$$

Note that $\kappa(P) \leq \underline{\psi}(P)$.

Lemma 3.4. *Let $c \geq 0$ and let P, R be two unbounded non-negative continuous piecewise linear functions defined on $[c, +\infty)$, with slopes 0 and 1, and which change from slope 1 to 0 infinitely many times. Suppose that $|\mathrm{P}(q) - \mathrm{R}(q)| = o(q)$ as q tends to infinity. Then, the non-increasing maps $\alpha \mapsto \kappa_\alpha(P)$ and $\alpha \mapsto \kappa_\alpha(\mathrm{R})$ have the same set of discontinuities on $[0, 1[$ and they coincide outside of this set. Moreover, we have $\kappa(P) = \kappa(\mathrm{R})$.*

Proof Let P and R be as above. By hypothesis we have $\overline{\psi}(P) = \overline{\psi}(\mathrm{R}) =: \overline{\psi}$ and $\underline{\psi}(P) = \underline{\psi}(\mathrm{R}) =: \underline{\psi}$. Fix $\alpha < \beta < \overline{\psi}$ and let us denote by $(q_i^P)_i$, $(q_{i,\beta}^P)_i$, $(r_{i,\beta}^P)_i$ (resp. $(q_i^R)_i$, $(q_{i,\alpha}^R)_i$, $(r_{i,\alpha}^R)_i$) the quantities associated by Definition 3.3 to $\kappa_\beta(P)$ (resp. $\kappa_\alpha(\mathrm{R})$). Let us first prove that

$$\kappa_\beta(P) \leq \kappa_\alpha(\mathrm{R}). \tag{3.4}$$

Let $\varepsilon > 0$ be such that $\alpha + \varepsilon < \beta$ and fix i arbitrarily large. If $\mathrm{R}(q) \geq \alpha q$ for each $q \in K_i := [q_{i,\alpha}^{\mathrm{R}}, q_{i+1,\alpha}^{\mathrm{R}}]$, then we set $r = s = r_{i,\alpha}^{\mathrm{R}}$. Otherwise $[r, s]$ denotes the maximal subinterval of K_i on which $R(q) \leq \alpha q$. Let us write $A_1 = (r, \mathrm{R}(r))$, $A_2 = (s, \mathrm{R}(s))$ and $A_3 = (r_{i,\alpha}^{\mathrm{R}}, \mathrm{R}(q_{i,\alpha}^{\mathrm{R}}))$. The graph of R above the interval K_i is contained inside the triangle $(A_1A_2A_3)$. Let us denote by $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) the horizontal line passing through the point $(r, \mathrm{R}(r) + \varepsilon r)$ (resp. the line with slope 1 passing through $(s, \mathrm{R}(s) + \varepsilon s)$ (see Figure 1)).

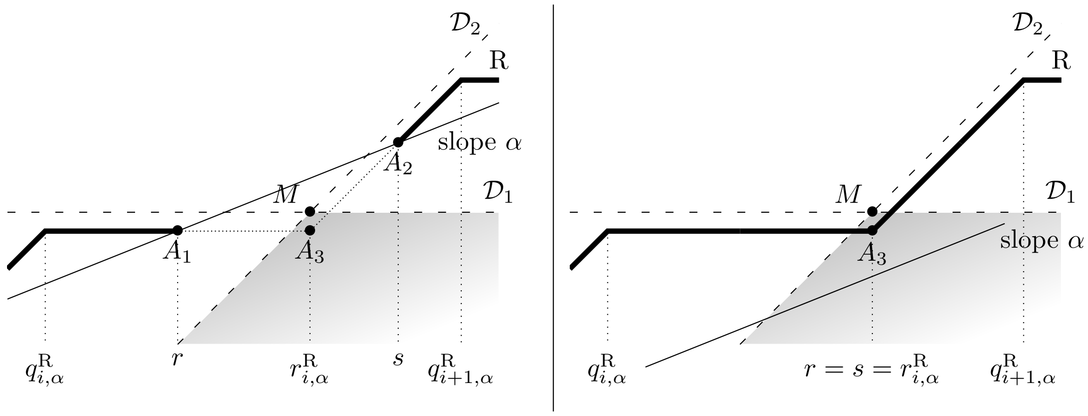

Figure 1: Graph of R on $[q_{i,\alpha}^{\mathrm{R}}, q_{i+1,\alpha}^{\mathrm{R}}]$

Now, let us define j as the maximal index such that $q_{j,\beta}^P \leq r$. The horizontal line passing through $(q_{j,\beta}^P, P(q_{j,\beta}^P))$ lies below the line $\mathcal{D}_1$. If $r = s$, then $q_{j+1,\beta}^P \geq s$. Otherwise, if i is large enough, then for each $q \in [r, s]$ we have:

$$P(q) \leq \mathrm{R}(q) + \varepsilon q \leq (\alpha + \varepsilon)q < \beta q,$$

which implies that $q_{j+1,\beta}^P \geq s$. It follows that the line with slope 1 passing through $(q_{j+1,\beta}^P, P(q_{j+1,\beta}^P))$ is below the line $\mathcal{D}_2$. As a consequence, the point $(r_{j,\beta}^P, P(q_{j,\beta}^P))$ lies in the area below $\mathcal{D}_1$ and $\mathcal{D}_2$ (the gray area of Figure 1). Let $M = (x_M, y_M)$ denote the intersection point of $\mathcal{D}_1$ and $\mathcal{D}_2$. Then we have

$$\frac{P(q_{j,\beta}^P)}{r_{j,\beta}^P} \leq \frac{y_M}{x_M} = \frac{\mathrm{R}(q_{i,\alpha}^{\mathrm{R}}) + \varepsilon r}{r_{i,\alpha}^{\mathrm{R}} - \varepsilon s} \leq \frac{\mathrm{R}(q_{i,\alpha}^{\mathrm{R}}) + \varepsilon r_{i,\alpha}^{\mathrm{R}}}{r_{i,\alpha}^{\mathrm{R}}} \cdot \Big(1 - \frac{\varepsilon}{1-\alpha}\Big)^{-1},$$

since $(1-\alpha)s \leq r_{i,\alpha}^{\mathrm{R}}$. By taking the infimum over i, we obtain

$$\kappa_\beta(P) \leq \big(\kappa_\alpha(\mathrm{R}) + \varepsilon\big) \cdot \Big(1 - \frac{\varepsilon}{1-\alpha}\Big)^{-1},$$

and by letting ε tend to 0 we prove (3.4). By symmetry, we also have $\kappa_\beta(\mathrm{R}) \leq \kappa_\alpha(P)$, which yields $\kappa_\alpha(P) = \kappa_\alpha(\mathrm{R})$ at each point where $\alpha \mapsto \kappa_\alpha(\mathrm{R})$ is continuous. By letting successively β and α tend to $\overline{\psi}$ in (3.4), it follows that $\kappa(P) \leq \kappa(\mathrm{R})$. By symmetry we have $\kappa(\mathrm{R}) \leq \kappa(P)$ and therefore $\kappa(P) = \kappa(\mathrm{R})$. □

Proposition 3.5. *Let* $\mathbf{u} = (1, \xi, \eta)$ *where* ξ, η *are non-zero real numbers and define the functions* L_i, L_i^* $(i = 1, 2, 3)$ *as in Section 3.1 (with respect to* $\mathbf{u}$*). Then*

$$\kappa(L_3) = -\kappa^*(L_1^*) = \frac{\widehat{\lambda}_{\min}(\xi, \eta)}{1 + \widehat{\lambda}_{\min}(\xi, \eta)}. \tag{3.5}$$

Proof Mahler's duality implies that $|L_3 + L_1^*|$ is bounded. By Lemma 3.4 we conclude that $\kappa(L_3) = -\kappa^*(L_1^*)$. Now, let us define N and Δ by $\mathrm{N}(\mathbf{x}) = \|\mathbf{x}\|$ and $\Delta = \|\mathbf{u} \wedge \mathbf{x}\|$ $(\mathbf{x} \in \mathbb{R}^3)$. For each $\mu \geq 0$ with $\mu < \lambda(\xi, \eta)$, we denote by $\widehat{\nu}_\mu(\xi, \eta)$ the exponent associated to N and Δ as in Section 2. Let $(\mathbf{x}_i)_{i \geq 0}$ be a sequence of minimal points associated to N and Δ and let us write $X_i := \|\mathbf{x}_i\|$, $\Delta_i := \|\mathbf{x}_i \wedge \mathbf{u}\|$ $(i \geq 0)$. A wellknown result in parametric geometry of numbers (see [14, §4]) states that:

$$L_1^*(q) = \min_{i \in \mathbb{N}} L_{\mathbf{x}_i}^*(q) \quad (q > 0), \tag{3.6}$$

where $L_{\mathbf{x}_i}^*(q) = \max\big(\log \Delta_i, \log X_i - q\big)$ (see (3.1)). Let us fix $0 \leq \alpha < \overline{\psi}(-L_1^*) = -\underline{\psi}_1^*$ and set $\mu := \alpha/(1 - \alpha) \geq 0$. Let us first prove that

$$\kappa_\alpha(-L_1^*) = \frac{\widehat{\nu}_\mu(\xi, \eta)}{1 + \widehat{\nu}_\mu(\xi, \eta)}. \tag{3.7}$$

We denote by $(q_i)_{i \geq 0}$, $(q_{k,\alpha})_k$ and $(r_{k,\alpha})_k$ the sequences associated to $\kappa_\alpha(-L_1^*)$ by Definition 3.3. Eq. (3.6) implies that q_i is the point at which $L_{\mathbf{x}_i}^*$ changes slope (from -1 to 0), which is precisely $\log(X_i) - \log(\Delta_i)$. Let $i_1 < i_2 < \dots$ denote the sequence of indices i such that $\Delta_i \leq \mathrm{X}_i^{-\mu}$. We claim that the sequence $(q_{k,\alpha})_k$ is the sequence $(q_{i_k})_k$. Indeed, the condition $-L_1^*(q_k)/q_k \geq \alpha$ is equivalent to the condition $\Delta_k \leq X_k^{-\mu}$, by using $-L_1^*(q_k) = -\log(\Delta_k)$. This implies that $r_{k,\alpha} = \log(X_{i_{k+1}}) - \log(\Delta_{i_k})$, and we thus have

$$\kappa_\alpha(-L_1^*) = \liminf_{k \to \infty} \frac{-L_1^*(q_{i_k})}{r_{k,\alpha}} = \liminf_{k \to \infty} \frac{-\log(\Delta_{i_k})}{\log(X_{i_{k+1}}) - \log(\Delta_{i_k})}. \tag{3.8}$$

Eqns. (2.2) and (3.8) together give (3.7). We conclude by noticing that when α tends to $-\underline{\psi}_1^* = \overline{\psi}_3$, then μ tends to $-\underline{\psi}_1^*/(1 + \underline{\psi}_1^*) = \lambda(\xi, \eta)$ by (3.3) and (3.2). □

4 Proof of Theorem 1.2

Recall that the first part of Theorem 1.2 follows from Laurent's inequalities (1.4) and from the fact that if $\lambda(\xi, \eta) = 1/2$, then $\widehat{\lambda}_{\min}(\xi, \eta) = 1/2$. Let us prove the second part of Theorem 1.2. Theorem 1.1 implies that there exist real numbers ξ, η with $1, \xi, \eta$ linearly independent over $\mathbb{Q}$, such that $\lambda(\xi, \eta) = \widehat{\lambda}_{\min}(\xi, \eta) = 1/2$. Now, let $\widehat{\lambda} \in \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{+\infty\}$ satisfying (1.3). The strategy of the proof is to construct a 3-system $\mathbf{P} = (P_1, P_2, P_3)$ such that

$$\lim_{q \to \infty} P_1(q) = +\infty, \quad \overline{\psi}(P_3) = \frac{\lambda}{1 + \lambda} \quad \text{and} \quad \kappa(P_3) = \frac{\widehat{\lambda}}{1 + \widehat{\lambda}}, \tag{4.1}$$

with the convention that $\lambda/(1 + \lambda) = 1$ if $\lambda = +\infty$. If $\mathbf{P}$ is as above, Theorem 3.1 gives a non-zero vector $\mathbf{u} \in \mathbb{R}^3$ such that $\|\mathbf{L_u} - \mathbf{P}\|$ is bounded. Moreover, we may suppose $\mathbf{u} = (1, \xi, \eta)$ with $1, \xi, \eta$ linearly independent over $\mathbb{Q}$, since P_1 is not bounded. Then, Lemma 3.4, Proposition 3.5 and relation (3.3) imply that $\lambda(\xi, \eta) = \lambda$ and $\widehat{\lambda}_{\min}(\xi, \eta) = \widehat{\lambda}$.
In order to cover the full joint spectrum of $(\lambda, \widehat{\lambda}_{\min})$ we distinguish between two cases. Our first construction deals with the case $\max(1 - \lambda, 0) < \widehat{\lambda}$ (note that this inequality is fulfilled if $\widehat{\lambda} \geq 1/2$,

since $\lambda > 1/2$) and the second one deals with the case $\widehat{\lambda} \le 1/2$.

First case. Suppose that $\lambda, \widehat{\lambda}$ satisfy $1 < \lambda + \widehat{\lambda}$ and $0 < \widehat{\lambda}$. For convenience, let us define $\nu \in (0, 1/\widehat{\lambda}]$ by $1/\nu = \widehat{\lambda}\big(1 + 1/\lambda\big)\big(1 + \widehat{\lambda}/\lambda\big)$. This number satisfies the relations

$$1 - \frac{\widehat{\lambda}}{\lambda}\nu - \Big(\frac{\widehat{\lambda}}{\lambda}\Big)^2 \nu = \frac{\lambda}{1+\lambda} \quad \text{and} \quad 1 + \nu - \Big(\frac{\widehat{\lambda}}{\lambda}\Big)^2 \nu = \frac{\lambda}{1+\lambda} \cdot \frac{1+\widehat{\lambda}}{\widehat{\lambda}}. \tag{4.2}$$

Under our hypotheses on λ and $\widehat{\lambda}$ we have

$$\frac{\widehat{\lambda}}{\lambda} < \frac{1}{\nu} - \frac{\widehat{\lambda}}{\lambda} - \Big(\frac{\widehat{\lambda}}{\lambda}\Big)^2 \le 1. \tag{4.3}$$

Indeed, the first inequality of (4.3) is equivalent to $1 < \lambda + \widehat{\lambda}$ and the second one is equivalent to the third inequality of (1.3). Let $(\beta_k)_{k \ge 0}$ be a non-decreasing sequence of real numbers > 1 such that β_k tends to $\lambda/\widehat{\lambda} \in (1, +\infty]$ as k tends to infinity. If $\lambda = +\infty$, we may take $\beta_k = k+1$ for each $k \ge 0$. If $\lambda < +\infty$, then we may simply take $\beta_k = \lambda/\widehat{\lambda}$ for each $k \ge 0$. Since $\lambda/\widehat{\lambda} > 1$, the sequence $(q_k)_{k \ge 0}$ defined by $q_k := \prod_{i=0}^{k} \beta_i$ tends to infinity. By (4.3) and by the choice of $(\beta_k)_k$, there is an index $N \ge 1$ such that for each $k \ge N$ we have:

$$\frac{1}{\beta_k \beta_{k-1}} \le \frac{1}{\beta_k} < \frac{1}{\nu} - \frac{1}{\beta_k} - \frac{1}{\beta_k \beta_{k-1}} \le 1. \tag{4.4}$$

For each $k \ge 1$, we define a point $\mathbf{a}^{(k)} = (a_1^{(k)}, a_2^{(k)}, a_3^{(k)}) \in \mathbb{R}^3$ by

$$\mathbf{a}^{(k)} = q_k \times \Big(\frac{\nu}{\beta_k \beta_{k-1}}, \frac{\nu}{\beta_k}, 1 - \frac{\nu}{\beta_k} - \frac{\nu}{\beta_k \beta_{k-1}} \Big).$$

Note that $a_1^{(k+1)} = a_2^{(k)}$ since $q_{k+1} = \beta_{k+1} q_k$, and that $a_1^{(k)} + a_2^{(k)} + a_3^{(k)} = q_k$. Inequalities (4.4) may be rewritten as $a_1^{(k)} \le a_2^{(k)} < a_3^{(k)} \le a_2^{(k+1)}$ for each $k \ge N$. We now construct the 3-system $\mathbf{P}$ on $[q_N, +\infty)$ whose combined graph is shown on figure 2.

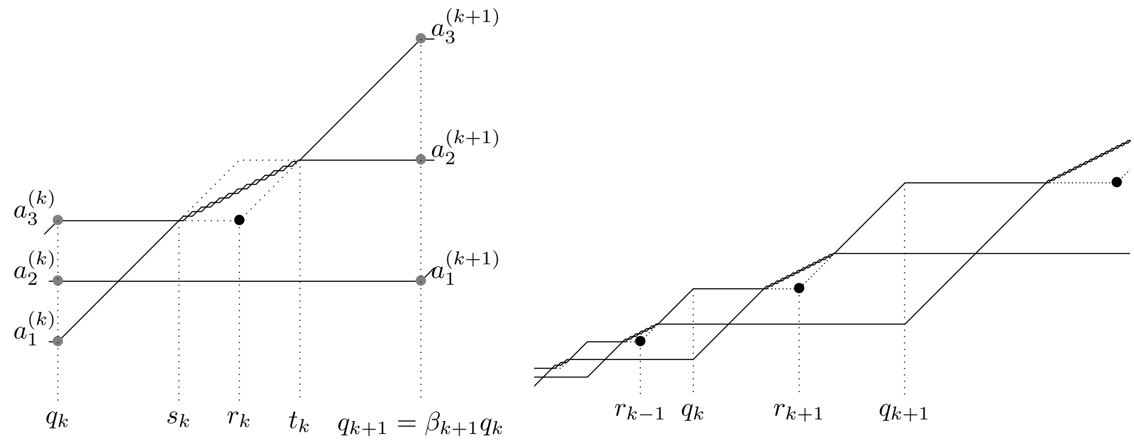

Figure 2: combined graph of a 3-system $\mathbf{P}$

Set $\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \, ; \, x_1 \le x_2 \le x_3\}$ and denote by $\Phi : \mathbb{R}^3 \to \Delta$ the continuous map which lists the coordinates of a point in monotone non-decreasing order. Let s_k and t_k be such that $a_1^{(k)} + s_k - q_k = a_3^{(k)}$ and $a_2^{(k+1)} = a_3^{(k+1)} - (q_{k+1} - t_k)$. We have

$$s_k = \Big(2 - \frac{\nu}{\beta_k} - \frac{2\nu}{\beta_k \beta_{k-1}}\Big) q_k \quad \text{and} \quad t_k = \Big(2\nu + \frac{\nu}{\beta_k}\Big) q_k,$$

and thus $s_k \le t_k$ thanks to the last inequality of (4.4). We define

$$\mathbf{P}(q) = \begin{cases} \Phi\big(a_1^{(k)} + q - q_k, a_2^{(k)}, a_3^{(k)}\big) & \text{if } q_k \le q \le s_k \\ \Phi\big(a_1^{(k+1)}, a_2^{(k+1)}, a_3^{(k+1)} + q - q_{k+1}\big) & \text{if } t_k \le q < q_{k+1}. \end{cases}$$

In order to define $\mathbf{P}$ on $[s_k, t_k]$, note that the ratio $a_2^{(k+1)}/t_k$ is smaller than $1/2$ and tends to $1/(2+\widehat{\lambda}/\lambda)$ as k tends to infinity, whereas the ratio $a_3^{(k+1)}/q_{k+1}$ tends to $\lambda/(1+\lambda)$ by using (4.2). Yet, the inequality $1 < \lambda + \widehat{\lambda}$ implies that the first limit is less than the second one. There exists therefore a real number θ such that

$$\frac{a_2^{(k+1)}}{t_k} < \theta < \frac{a_3^{(k+1)}}{q_{k+1}}$$

for k large enough. For each $q \in [s_k, t_k]$, we set $P_1(q) = a_2^{(k)}$ and we define $P_2(q)$ and $P_3(q)$ such that $\mathbf{P} = (P_1, P_2, P_3)$ satisfies the hypotheses of a 3-system (which is possible since the line passing through the points $(s_k, a_3^{(k)})$ and $(t_k, a_2^{(k+1)})$ has slope $1/2$) and such that, when k is large enough, we have

$$\frac{P_3(q)}{q} < \theta \quad \text{for each } q \in [s_k, t_k] \tag{4.5}$$

(see figure 2). Let r_k be the abscissa of the intersection of the horizontal line passing through $(q_k, P_3(q_k))$ and of the line with slope 1 passing through $(q_{k+1}, P_3(q_{k+1}))$. We have

$$r_k = a_3^{(k)} - a_3^{(k+1)} + q_{k+1} = \Big(1 + \nu - \frac{\nu}{\beta_k\beta_{k-1}}\Big) q_k.$$

By (4.5) and (4.2), for each $\theta \le \alpha < \lambda/(1+\lambda)$ we have

$$\overline{\psi}(P_3) = \limsup_{k\to\infty} \frac{P_3(q_k)}{q_k} = \limsup_{k\to\infty} \frac{a_3^{(k)}}{q_k} = \frac{\lambda}{1+\lambda} \quad \text{and} \quad \kappa_\alpha(P_3) = \kappa(P_3) = \liminf_{k\to\infty} \frac{P_3(q_k)}{r_k} = \frac{\widehat{\lambda}}{1+\widehat{\lambda}}.$$

Thus, $\mathbf{P}$ satisfies (4.1), which concludes the first case.

Second case. Suppose that $\widehat{\lambda} \le 1/2$. Under this additional condition, (1.3) may simply be rewritten as $0 \le \widehat{\lambda} \le \frac{1}{2} < \lambda \le +\infty$, which is equivalent to

$$0 \le \frac{\widehat{\lambda}}{1+\widehat{\lambda}} \le \frac{1}{3} < \frac{\lambda}{1+\lambda} \le 1.$$

Fix $\theta \in \mathbb{R}$ such that $1/3 < \theta < \lambda/(1+\lambda)$. Let $(\alpha_k)_{k\ge1}$, $(\psi_k)_{k\ge1}$ be two sequences of real numbers which tend to $\widehat{\lambda}/(1+\widehat{\lambda})$ and $\lambda/(1+\lambda)$ respectively, and such that for each $k \ge 1$, we have

$$0 < \alpha_k \le \frac{1}{3} < \theta < \psi_k < 1.$$

Let $(q_k)_{k\ge0}$ be the sequence defined by $q_0 = 1$ and

$$q_{k+1} = \frac{\psi_k}{1-\psi_{k+1}} \Big(\frac{1}{\alpha_k} - 1\Big) q_k \quad (k \ge 0). \tag{4.6}$$

Note that $q_{k+1}/q_k > 2\theta/(1-\theta) > 1$ for each $k \ge 0$, which implies that the sequence $(q_k)_k$ tends to infinity. For each $k \ge 0$, let us define the abscissas s_k and t_k by $s_k/3 = \psi_k q_k$ and $t_k/3 = (1-\psi_{k+1})q_{k+1}/2$. Let r_k be the abscissa of the intersection point of the horizontal line passing through $(q_k, \psi_k q_k)$ and of the line with slope 1 passing through $(q_{k+1}, \psi_{k+1} q_{k+1})$ (see figure 3). We have $r_k = \psi_k q_k + (1-\psi_{k+1})q_{k+1}$, which may be rewritten as $\alpha_k r_k = \psi_k q_k$ by (4.6). Since $0 < \alpha_k \leq 1/3$, we have $s_k \leq r_k \leq t_k$. Now, let $\mathbf{P} = (P_1, P_2, P_3)$ be a 3-system on $[q_0, +\infty[$ such that for each $k \geq 0$, we have

$$\frac{\mathbf{P}(q_k)}{q_k} = \Big(\frac{1-\psi_k}{2}, \frac{1-\psi_k}{2}, \psi_k\Big), \quad \frac{\mathbf{P}(s_k)}{s_k} = \frac{\mathbf{P}(t_k)}{t_k} = \Big(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\Big),$$

and such that

$$\frac{\mathbf{P}(q)}{q} \leq \theta \quad \text{for each } q \in [s_k, t_k]. \tag{4.7}$$

An example of such 3-system is represented on figure 3.

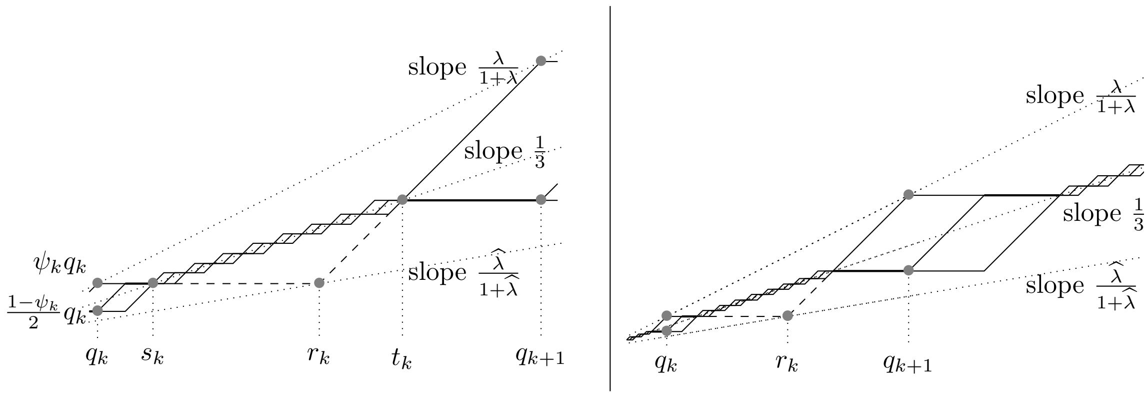

Figure 3: combined graph of the 3-system $\mathbf{P}$

By (4.7) and since $\theta \leq \psi_k$ for each $k \geq 0$, it is clear that such a 3-system $\mathbf{P}$ satisfies $\overline{\psi}(P_3) = \lambda/(1+\lambda)$. Moreover, (4.7) also implies that $\kappa_\alpha(P_3) = \widehat{\lambda}/(1+\widehat{\lambda})$ for each α such that $\theta < \alpha < \lambda/(1+\lambda)$. We thus have $\kappa(P_3) = \widehat{\lambda}/(1+\widehat{\lambda})$ and $\mathbf{P}$ satisfies (4.1). This ends the proof of Theorem 1.2.

References

[1] Y. Bugeaud. Exponents of Diophantine approximation. In D. Badziahin, A. Gorodnik, and N. Peyerimhoff, editors, *Dynamics and Analytic Number Theory*, volume 437 of *London Math. Soc. Lecture Note Ser.*, chapter 2, pages 96–135. Cambridge University Press, 2016.

[2] Y. Bugeaud and M. Laurent. Exponents of Diophantine approximation and Sturmian continued fractions. *Ann. Inst. Fourier*, 55(3):773–804, 2005.

[3] Y. Bugeaud and M. Laurent. On exponents of homogeneous and inhomogeneous Diophantine approximation. *Moscow Math. J*, 5(4):747–766, 2005.

[4] J. Cassaigne. Limit values of the recurrence quotient of Sturmian sequences. *Theoret. Comput. Sci.*, 218(1):3–12, 1999.

[5] H. Davenport and W. Schmidt. Approximation to real numbers by quadratic irrationals. *Acta Arith.*, 13(2):169–176, 1967.

[6] H. Davenport and W. Schmidt. Approximation to real numbers by algebraic integers. *Acta Arith.*, 15(4):393–416, 1969.

[7] S. Fischler. Palindromic prefixes and diophantine approximation. *Monatsh. Math.*, 151(1):11–37, 2007.

[8] M. Laurent. Exponents of Diophantine approximation in dimension two. *Canad. J. Math*, 61:165–189, 2009.

[9] A. Poëls. Exponents of Diophantine approximation in dimension 2 for numbers of Sturmian type. *arXiv preprint arXiv:1711.07896*, 2017.

[10] A. Poëls. *Applications de la géométrie paramétrique des nombres à l'approximation diophantienne*. PhD thesis, Université Paris-Sud, https://tel.archives-ouvertes.fr/tel-01827304/document, 2018.

[11] D. Roy. On Schmidt and Summerer parametric geometry of numbers. *Ann. of Math.*, 182:739–786, 2015.

[12] D. Roy. Spectrum of the exponents of best rational approximation. *Math. Z.*, 283(1-2):143–155, 2016.

[13] J. Schleischitz. On the spectrum of Diophantine approximation constants. *Mathematika*, 62(1):79–100, 2016.

[14] W. M. Schmidt and L. Summerer. Parametric geometry of numbers and applications. *Acta Arith.*, 140:67–91, 2009.

[15] W. M. Schmidt and L. Summerer. Diophantine approximation and parametric geometry of numbers. *Monatsh. Math.*, 169:51–104, 2013.

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