On approximation to a real number by algebraic numbers of bounded degree

Anthony Poëls

Abstract

In his seminal 1961 paper, Wirsing studied how well a given transcendental real number ξ can be approximated by algebraic numbers α of degree at most n for a given positive integer n, in terms of the so-called naive height $H(\alpha)$ of α. He showed that the supremum $\omega_n^*(\xi)$ of all ω for which infinitely many such α have $|\xi-\alpha| \le H(\alpha)^{-\omega-1}$ is at least $(n+1)/2$. He also asked if we could even have $\omega_n^*(\xi) \ge n$ as it is generally expected. Since then, all improvements on Wirsing's lower bound were of the form $n/2+\mathcal{O}(1)$ until Badziahin and Schleischitz showed in 2021 that $\omega_n^*(\xi) \ge an$ for each $n \ge 4$, with $a = 1/\sqrt{3} \simeq 0.577$. In this paper, we use a different approach partly inspired by parametric geometry of numbers and show that $\omega_n^*(\xi) \ge an$ for each $n \ge 2$, with $a = 1/(2-\log 2) \simeq 0.765$.

MSC 2020: 11J82 (Primary), 11J13 (Secondary).

Keywords: Wirsing's conjecture, approximation by algebraic numbers, polynomial approximation, exponents of approximation, transcendance.

1 Introduction

One of the fundamental questions in Diophantine approximation is the following. Given an irrational real number ξ, how well can it be approximated by rational numbers? It follows from the theory of continued fractions that there exist infinitely many rational numbers p/q with $q \ge 1$ and

$$\left|\xi - \frac{p}{q}\right| \le \frac{1}{q^2}. \tag{1.1}$$

The above property is optimal in the following sense. For any fixed $\varepsilon > 0$, the set of real numbers ξ for which there exist infinitely many p/q with $|\xi - p/q| \le 1/q^{2+\varepsilon}$ has Lebesgue measure zero. This is an easy application of the Borel-Cantelli Lemma (see [5, Corollary 1.5], also see [3, §23]). If we think of rational numbers as algebraic numbers of degree one, then it is natural to generalize the previous question in the following way: given a positive integer n, how well can ξ be approximated by algebraic numbers of degree at most n? In [11] Koksma introduced a classification of real numbers in terms of the behaviour of the sequence $(\omega_n^*(\xi))_{n\ge1}$, where the classical exponent $\omega_n^*(\xi)$ is defined as the supremum of the real numbers $\omega^* > 0$ for which the inequalities

$$0 < |\xi - \alpha| \le H(\alpha)^{-\omega^*-1} \tag{1.2}$$

admit infinitely many solutions in algebraic numbers α of degree at most n. Here, $H(\alpha)$ denotes the (naive) *height* of α, that is the largest absolute value of the coefficients of its irreducible polynomial over $\mathbb{Z}$. See Section 2 for a motivation of the summand -1 appearing in the exponent in (1.2). By a result of Sprindžuk [17] combined with classical transference inequalities (see [14, Chapter VIII, Section 9] and [6, Theorem 2.8]), we have

$$\omega_n^*(\xi) = n \tag{1.3}$$

for almost all real numbers ξ with respect to Lebesgue measure. Schmidt's Subspace theorem implies that (1.3) also holds if ξ is algebraic of degree $\geq n+1$ (see [14, Chapter 6, Corollary 1E]). However, given a specific transcendental real number ξ, it is usually extremely difficult to determine $\omega_n^*(\xi)$. We can find in Wirsing's original 1961 paper [20] the following famous problem, which is the main motivation for the present work.

Wirsing's problem. Do we have $\omega_n^*(\xi) \geq n$ for any integer $n \geq 1$ and any transcendental real number ξ?

So far, and despite a lot of effort, it has been confirmed only for $n = 1$ (this is a consequence of (1.1)) and for $n = 2$ (by Davenport and Schmidt [7], also see [8]). In his 1961 paper, Wirsing proved

$$\omega_n^*(\xi) \geq \frac{n+1}{2}, \tag{1.4}$$

valid for each transcendental real number ξ. This was the first lower bound for $\omega_n^*(\xi)$ in terms of n only. Until very recently, the best lower bounds due to Bernik and Tishchenko [2, 18, 19] were of the form $n/2 + \mathcal{O}(1)$. In 2021, Badziahin and Schleischitz made an important breakthrough [1] by improving on the factor $1/2$ for the first time. More precisely, they showed that for each $n \geq 4$ and each transcendental real number ξ, we have

$$\omega_n^*(\xi) \geq an, \qquad \text{where } a = \frac{1}{\sqrt{3}} = 0.577\cdots$$

Our main result improves the above result as follows.

Theorem 1.1. *Let n be an integer ≥ 2. For any transcendental real number ξ, we have*

$$\omega_n^*(\xi) \geq an, \qquad \textit{where } a = \frac{1}{2-\log 2} = 0.765\cdots.$$

Note that our bounds are better than those obtained in [19] starting with $n = 7$. We believe that the constant a in Theorem 1.1 is not optimal and could be improved by refining our method.

Given a transcendental real number ξ, Wirsing's approach to showing his lower bound (1.4) is to construct coprime polynomials P and Q of degree at most n, which have integer coefficients and have very small absolute values at ξ. Considering their resultant, he then proves that a root of P or Q must be very close to ξ. For the proof of Theorem 1.1, the key-point is to consider simultaneously $n+1$ linearly independent polynomials $P_1, \ldots, P_{n+1} \in \mathbb{Z}[X]_{\leq n}$, instead of just two. This idea has its origins in [12], where we improve the upper bound for the uniform exponent of polynomial approximation.

This paper is organized as follows. In Sections 3 and 4, we construct the aforementioned polynomials P_i, which roughly realize the successive minima of a certain symmetric convex body in $\mathbb{R}[X]_{\leq n}$ (with respect to the lattice of integer polynomials $\mathbb{Z}[X]_{\leq n}$). We are able to control rather precisely their size and their absolute value at ξ. In Section 5, by evaluating some kind of non-zero generalized resultant, we prove that for each $k = 2, \dots, n+1$, at least one of the polynomials $P_1, \dots, P_k$ has a root very close to ξ. Taking into account all these approximations, we then conclude in Section 6 that $\omega_n^*(\xi)$ is bounded below by the minimum of an explicit function of $n+1$ variables. In the last two Sections 7 and 8, which are independent from the previous ones, we deal with the optimization problem of finding this minimum. We show that it is at least equal to $n/(2 - \log(2))$.

2 Notation

For any functions $f, g : I \to [0, +\infty)$ on a set I, we write $f = \mathcal{O}(g)$ or $f \ll g$ or $g \gg f$ to mean that there is a positive constant c such that $f(x) \leq cg(x)$ for each $x \in I$. We write $f \asymp g$ when both $f \ll g$ and $g \ll f$ hold.

Given a ring A (typically $A = \mathbb{R}$ or $\mathbb{Z}$) and an integer $n \geq 0$, we denote by $A[X]$ the ring of polynomials in X with coefficients in A, and by $A[X]_{\leq n} \subseteq A[X]$ the subgroup of polynomials of degree at most n. We say that $P \in \mathbb{Z}[X]$ is *primitive* if it is non-zero and the greatest common divisor of its coefficients is 1. Given $P(X) = \sum_{k=0}^{n} a_k X^k \in \mathbb{R}[X]$, we set

$$\|P\| = \max_{0 \leq k \leq n} |a_k|.$$

For $k = 0, \dots, n$, we define

$$P^{[k]} = \frac{1}{k!} \frac{d^k P}{dX^k} \in \mathbb{R}[X].$$

Then, for each real number ξ, we have

$$P(X) = \sum_{k=0}^{n} P^{[k]}(\xi)(X - \xi)^k. \tag{2.1}$$

We denote by $\det(P_1, \dots, P_{n+1})$ the determinant of a family of $n+1$ polynomials $P_1, \dots, P_{n+1}$ in $\mathbb{R}[X]_{\leq n}$ with respect to the canonical basis $(1, X, X^2, \cdots, X^n)$ of $\mathbb{R}[X]_{\leq n}$. Note that the change-of-basis matrix from the canonical basis to the basis $(1, X - \xi, (X - \xi)^2, \dots, (X - \xi)^n)$ has determinant 1. Using (2.1), it follows that

$$\det(P_1, \dots, P_{n+1}) = \det\left(P_j^{[i-1]}(0)\right)_{1 \leq i,j \leq n+1} = \det\left(P_j^{[i-1]}(\xi)\right)_{1 \leq i,j \leq n+1}. \tag{2.2}$$

For short, we say that polynomials of $\mathbb{R}[X]_{\leq n}$ or $\mathbb{Z}[X]_{\leq n}$ are *linearly independent* to mean that they are linearly independent over $\mathbb{R}$. We identify $\mathbb{R}^{n+1}$ to $\mathbb{R}[X]_{\leq n}$ via the isomorphism

$$(a_0, \dots, a_n) \longmapsto a_0 + a_1 X + \cdots + a_n X^n.$$

Then, the volume $\mathrm{vol}(C)$ of a closed set $C \subseteq \mathbb{R}[X]_{\leq n}$ is simply the Lebesgue measure of the corresponding set in $\mathbb{R}^{n+1}$. Given $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{R}^{n+1}$, we also write

$$\|\mathbf{a}\| = \max_{0 \leq k \leq n} |a_k|.$$

Let $\xi \in \mathbb{R}$ be a transcendental number and n a positive integer. The following two classical Diophantine exponents will play an important role in our study. We denote by $\widehat{\omega}_n(\xi)$ (resp. $\omega_n(\xi)$), the supremum of the real numbers $\omega > 0$ such that the system

$$\|P\| \leq H \quad \text{and} \quad 0 < |P(\xi)| \leq H^{-\omega}$$

admits a non-zero solution $P \in \mathbb{Z}[X]_{\leq n}$ for each large enough H (resp. for arbitrarily large H). Dirichlet's Theorem implies that

$$n \leq \widehat{\omega}_n(\xi) \leq \omega_n(\xi) \tag{2.3}$$

(see for example [14, Chapter 2, Theorem 1C]). The exponent $\omega_n^*(\xi)$, as defined in the introduction, is the supremum of the real numbers $\omega^* > 0$ for which there are infinitely many algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| \leq H(\alpha)^{-\omega^* - 1}. \tag{2.4}$$

Here, $H(\alpha) = \|P_\alpha\|$, where P_α is the minimal polynomial of α irreducible over $\mathbb{Z}$ (with positive leading coefficient). The reader may consult [6] for an interesting survey presenting, among others, several transference inequalities between the exponents $\omega_n(\xi)$, $\widehat{\omega}_n(\xi)$ and $\omega_n^*(\xi)$. In particular we have the inequality

$$\omega_n^*(\xi) \leq \omega_n(\xi),$$

whose simplicity is a consequence of having the summand -1 in the exponent of $H(\alpha)$ in (2.4) (see [6, Theorem 2.5]). According to [6, Theorems 2.6 and 3.1], we further have

$$\omega_n^*(\xi) \geq \omega_n(\xi) - n + 1.$$

Thus, if $\omega_n(\xi) = \infty$, then $\omega_n^*(\xi) = \infty \geq n$ and our main Theorem 1.1 holds for ξ. Consequently, starting with Section 4 we will assume that

$$\omega_n(\xi) < \infty.$$

We also recall the well-known fact that any non-constant polynomial $P \in \mathbb{Z}[X]_{\leq n}$ has a root α satisfying

$$|\xi - \alpha||P'(\xi)| \leq n|P(\xi)|, \tag{2.5}$$

(a formula which is easily proved using the logarithmic derivative of P, see [6, Section 2]). By Gelfond's Lemma (see *e.g.* [5, Lemma A.3] as well as [4]), for each non-zero $P, Q \in \mathbb{Z}[X]_{\leq n}$, if P divides Q, then

$$e^{-n}\|P\| < \|Q\|. \tag{2.6}$$

In particular, if $\|Q\| \leq e^{-n}\|P\|$, then P cannot be a factor of Q. Moreover, if α is a root of P, then $H(\alpha) \leq e^n\|P\|$ since P_α divides P.

Heuristically, one expects the existence of infinitely many non-constant $P \in \mathbb{Z}[X]_{\leq n}$ with $|P(\xi)| \ll \|P\|^{-n}$ and $|P'(\xi)| \asymp \|P\|$ (the last estimate holds unless P has two roots close to ξ). Then, by (2.5), each such polynomial has a root α with $|\xi - \alpha| \ll \|P\|^{-n-1} \ll H(\alpha)^{-n-1}$ which is a strong form of the conjecture since, in view of the definition of $\omega_n^*(\xi)$ in (2.4), it implies that $\omega_n^*(\xi) \geq n$.

Finally, if $\mathcal{A}$ is a subset of a $\mathbb{R}$-vector space V, we denote by $\langle \mathcal{A} \rangle_{\mathbb{R}} \subseteq V$ the $\mathbb{R}$-vector space spanned by $\mathcal{A}$, with the convention that $\langle \emptyset \rangle_{\mathbb{R}} = \{0\}$.

3 Parametric geometry of numbers

Let ξ be a transcendental real number and n be an integer ≥ 2. Schmidt and Summerer's parametric geometry of numbers [15, 16], [13] is a powerful tool for studying Diophantine exponents. Although we do not need much of this theory, it provides a convenient framework to state the results we will use. In this section we first recall some basic elements from parametric geometry of numbers, then we establish several lemmas which form the basis of our future polynomial constructions.

Following the approach of Roy [13] (with the maximum norm instead of the Euclidean norm), we consider for any parameter $q \geq 0$ the symmetric convex body

$$\mathcal{C}_\xi(q) = \Big\{ P \in \mathbb{R}[X]_{\leq n}\,;\, \|P\| \leq 1 \quad \text{and} \quad |P(\xi)| \leq e^{-q} \Big\}.$$

For $i = 1, \dots, n+1$, we define $L_i(q)$ as the smallest real number L such that $e^L\mathcal{C}_\xi(q) \cap \mathbb{Z}[X]_{\leq n}$ contains at least i linearly independent polynomials. Thus, $e^{L_1(q)}, \dots, e^{L_{n+1}(q)}$ are the successive minima of $\mathcal{C}_\xi(q)$ with respect to the lattice $\mathbb{Z}[X]_{\leq n}$. We group these minima in a map $\mathrm{L}_\xi : [0,\infty) \to \mathbb{R}^{n+1}$ defined by

$$\mathrm{L}_\xi(q) = \big(L_1(q), \cdots, L_{n+1}(q)\big).$$

Recall that the functions L_i are continuous, piecewise linear with slopes 0 and 1 (they are therefore non-decreasing). Furthermore, since $\mathrm{vol}\big(\mathcal{C}_\xi(q)\big) \asymp e^{-q}$, Minkowski's second theorem implies that

$$L_1(q) + \cdots + L_{n+1}(q) = q + \mathcal{O}(1), \qquad q \in [0,\infty),$$

where the implicit constant depends on n and ξ only. To any non-zero polynomial $P \in \mathbb{Z}[X]_{\leq n}$ we associate a function $L(P,\cdot) \to [0,+\infty)$ by setting

$$L(P,q) = \max\big\{ \log\|P\|, q + \log|P(\xi)| \big\} \qquad (q \in [0,+\infty)).$$

Following Roy's terminology [13, §2.2], the *trajectory* of a non-zero polynomial $P \in \mathbb{Z}[X]_{\le n}$ is the graph of the function $L(P,\cdot)$. Note that $L(P,\cdot)$ is continuous, piecewise linear, constant on $[0,q_P]$ and increasing with slope 1 on $[q_P,\infty)$, where the slope change point q_P is

$$q_P = \log\|P\| - \log|P(\xi)|.$$

Thus, for each $q \ge 0$, we have

$$L(P,q) = \begin{cases} \log\|P\| & \text{if } q \le q_P, \\ q + \log|P(\xi)| & \text{if } q \ge q_P. \end{cases}$$

Since, for each $q \ge 0$, the smallest $L \ge 0$ such that $P \in e^L \mathcal{C}_\xi(q)$ is precisely $L(P,q)$, we have

$$L_1(q) = \min_{P \in \mathbb{Z}[X]_{\le n}\setminus\{0\}} L(P,q). \tag{3.1}$$

Moreover, since ξ is transcendental, we have $\lim_{q\to\infty} L_1(q) = \infty$. Although we will not need them, we have the classical formulas (arguing as in [15, Theorem 1.4])

$$\underline{\varphi} = \liminf_{q\to\infty} \frac{L_1(q)}{q} = \frac{1}{1+\omega_n(\xi)} \quad \text{and} \quad \overline{\varphi} = \limsup_{q\to\infty} \frac{L_1(q)}{q} = \frac{1}{1+\widehat{\omega}_n(\xi)}.$$

The exponents $\underline{\varphi}$ and $\overline{\varphi}$ are parametric versions of the exponents $\omega_n(\xi)$ and $\widehat{\omega}_n(\xi)$.

Lemma 3.1. *Fix $\widehat{\omega} < \widehat{\omega}_n(\xi)$. There exists $q_0 = q_0(\widehat{\omega}) \ge 0$ with the following property. Let $q \in [q_0,\infty)$ and $Q \in \mathbb{Z}[X]_{\le n}$ be such that $L(Q,\cdot)$ has slope 1 on $[q,\infty)$ and coincides with L_1 at q. Then*

$$|Q(\xi)| \le e^{-\widehat{\omega} L_1(q)}. \tag{3.2}$$

Proof. Choose $q \ge 0$ and $Q \in \mathbb{Z}[X]_{\le n}$ such that $L(Q,\cdot)$ has slope 1 on $[q,\infty)$. This means that $L(Q,q) = q + \log|Q(\xi)|$. We also assume that $L(Q,q) = L_1(q)$ and set $H = e^{L_1(q)}$. By definition of $\widehat{\omega}_n(\xi)$, if q is large enough, there exists a non-zero $P \in \mathbb{Z}[X]_{\le n}$ such that

$$\|P\| < H = e^{L_1(q)} \quad \text{and} \quad |P(\xi)| \le H^{-\widehat{\omega}}.$$

Since $L_1(q) \le L(P,q) = \max\{\log\|P\|, q + \log|P(\xi)|\}$, this yields $L(P,q) = q + \log|P(\xi)|$, and

$$q + \log|Q(\xi)| = L_1(q) \le q + \log|P(\xi)| \le q - \widehat{\omega} L_1(q),$$

hence $\log|Q(\xi)| \le -\widehat{\omega} L_1(q)$, which is equivalent to (3.2). □

Lemma 3.2. *There exists a constant $c > 0$ which depends on n and ξ only such that, for any linearly independent polynomials $P_1,\dots,P_{n+1} \in \mathbb{Z}[X]_{\le n}$, we have*

$$1 \le c\|P_1\|\cdots\|P_{n+1}\| \sum_{i=1}^{n+1} \frac{|P_i(\xi)|}{\|P_i\|}.$$

Proof. Since $\det(P_1,\dots,P_{n+1})$ is a non-zero integer, it follows from (2.2) that

$$1 \le |\det(P_1,\dots,P_{n+1})| = \left|\det\left(P_j^{[i-1]}(\xi)\right)_{1\le i,j\le n+1}\right|.$$

We conclude by expanding the last determinant and by noting that for $j = 1,\dots,n+1$, we have $P_j^{[0]}(\xi) = P_j(\xi)$ and $|P_j^{[i-1]}(\xi)| \ll \|P_j\|$ $(i = 2,\dots,n+1)$. □

The following result is crucial for our approach. Under some condition, it provides $n+1$ linearly independent polynomials with integer coefficients which have "good" properties: their absolute values are small at ξ and their height are under control. In some way, it is reminiscent of [13, Theorem 3.1]. The idea is to start with a family of polynomials which realize the successive minima of $\mathcal{C}_\xi(q)$, and then to correct these polynomials to make their absolute values small at ξ.

Lemma 3.3. *Let $q \in [0,\infty)$ and $Q \in \mathbb{Z}[X]_{\le n}$ such that $L_1(q) = L(Q,q)$. We suppose that $L(Q,\cdot)$ has slope 1 on $[q,+\infty)$. Then, there exist linearly independent polynomials $P_1,\dots,P_{n+1} \in \mathbb{Z}[X]_{\le n}$ such that $P_1 = Q$ and*

(i) $|P_i(\xi)| < |P_1(\xi)|$ *and* $e^{L_i(q)} \le \|P_i\| \le 2e^{L_i(q)}$ *for* $i = 2,\dots,n+1$;

(ii) $\|P_1\| \le \cdots \le \|P_{n+1}\|$;

(iii) $|P_1(\xi)|\cdot\|P_2\|\cdots\|P_{n+1}\| \asymp 1$, *with implicit constants depending only on n and ξ.*

Proof. Let $Q_1 = Q, Q_2,\dots,Q_{n+1} \in \mathbb{Z}[X]_{\le n}$ be linearly independent polynomials which realize $L_1(q),\cdots,L_{n+1}(q)$, *i.e.* such that

$$L(Q_i,q) = L_i(q) \qquad (i = 1,\dots,n+1).$$

By hypothesis on $Q = Q_1$, we have $q \ge q_1$, where $q_1 = \log\|Q_1\| - \log|Q_1(\xi)|$ is the abscissa where $L(Q_1,\cdot)$ changes slope. We obtain

$$L_1(q) = L(Q_1,q) = \log\|Q_1\| + q - q_1 \quad \text{and} \quad \log|Q_1(\xi)| = L_1(q) - q.$$

Then, Minkowski's second theorem yields

$$|Q_1(\xi)|e^{L_2(q)+\cdots+L_{n+1}(q)} = e^{-q+L_1(q)+\cdots+L_{n+1}(q)} \ll 1. \tag{3.3}$$

Set $P_1 = Q_1$, and for $i = 2,\dots,n+1$, put

$$R_i = Q_i - \left\lfloor \frac{Q_i(\xi)}{P_1(\xi)} \right\rfloor P_1 \in \mathbb{Z}[X]_{\le n}.$$

We have $|R_i(\xi)| < |P_1(\xi)|$ and since $q_1 = \log\left(\|Q_1\|/|Q_1(\xi)|\right) = \log\left(\|P_1\|/|P_1(\xi)|\right)$, we also have

$$\begin{aligned}\|R_i\| \le 2\max\left\{\|Q_i\|, \frac{|Q_i(\xi)|}{|P_1(\xi)|}\cdot\|P_1\|\right\} &= 2\exp\Big(\max\big\{\log\|Q_i\|, \log|Q_i(\xi)| + q_1\big\}\Big)\\ &= 2e^{L(Q_i,q_1)}\\ &\le 2e^{L(Q_i,q)} = 2e^{L_i(q)}.\end{aligned}$$

Denote by $P_2, \dots, P_{n+1}$ the polynomials $R_2, \dots, R_{n+1}$ reordered by increasing norm. By the above, for $i = 2, \dots, n+1$, we have

$$|P_i(\xi)| < |P_1(\xi)| \quad \text{and} \quad \|P_i\| \leq 2e^{L_i(q)}. \tag{3.4}$$

On the other hand, since $\log |P_i(\xi)| + q < \log |P_1(\xi)| + q = L_1(q) \leq L(P_i, q)$ (the last inequality coming from the minimality property of (3.1)), we must have $L(P_i, q) = \log \|P_i\|$, thus

$$\log \|P_1\| \leq L(P_1, q) = L_1(q) \leq L(P_i, q) = \log \|P_i\|.$$

So, we have

$$L(P_1, q) \leq L(P_2, q) = \log \|P_2\| \leq \cdots \leq \log \|P_{n+1}\| = L(P_{n+1}, q).$$

Since the polynomials $P_1, P_2, \dots, P_{n+1} \in \mathbb{Z}[X]_{\leq n}$ are linearly independent, we deduce that

$$L_i(q) \leq L(P_i, q) = \log \|P_i\| \qquad \text{for } i = 2, \dots, n+1. \tag{3.5}$$

So the conditions (i) and (ii) are fulfilled. Finally, Lemma 3.2 together with (3.4) and (3.3) yields

$$1 \ll \prod_{i=1}^{n+1} \|P_i\| \sum_{i=1}^{n+1} \frac{|P_i(\xi)|}{\|P_i\|} \ll |P_1(\xi)| \prod_{i=2}^{n+1} \|P_i\| \ll |P_1(\xi)| e^{L_2(q)+\cdots+L_{n+1}(q)} \ll 1.$$

Note that (3.5) together with (3.4) show that $L_i(q) \leq L(P_i, q) \leq L_i(q) + \log 2$ for $i = 1, \dots, n+1$, while $L_1(q) = L(P_1, q)$. Thus, roughly speaking, the polynomials P_i realize the successive minima of $\mathcal{C}_\xi(q)$ up to a factor ≤ 2. □

4 Families of polynomials

From now on, we fix an integer $n \geq 2$ and a transcendental real number ξ with $\omega_n(\xi) < \infty$, which is no restriction for the proof of Theorem 1.1 as we saw in Section 2. In this section, we adjust our polynomial construction of Lemma 3.3 taking into account the exponents $\omega_n(\xi)$ and $\widehat{\omega}_n(\xi)$. Fix a small $\varepsilon \in (0, 1)$ and set

$$\widehat{\omega} = \widehat{\omega}(\varepsilon) = \widehat{\omega}_n(\xi) - \frac{\varepsilon}{2} \quad \text{and} \quad \omega = \omega(\varepsilon) = \omega_n(\xi) - \frac{\varepsilon}{2}. \tag{4.1}$$

It follows from the definition of $\omega_n(\xi)$ and $\widehat{\omega}_n(\xi)$ that there exists $H_0 \geq 1$ such that for each $H > H_0$, the system

$$\|Q\| \leq H \quad \text{and} \quad |Q(\xi)| \leq H^{-\widehat{\omega}} \tag{4.2}$$

has a non-zero solution $Q \in \mathbb{Z}[X]_{\leq n}$, and that any such Q satisfies

$$|Q(\xi)| \geq \|Q\|^{-\omega_n(\xi)-\varepsilon/2} \tag{4.3}$$

(because when H goes to infinity, the quantity $|Q(\xi)|$ tends to 0, and thus $\|Q\|$ also goes to infinity). Define

$$\mathcal{P}(\varepsilon) = \left\{ P \in \mathbb{Z}[X]_{\leq n} \text{ irreducible } ; \, e^{-n}\|P\| \geq H_0 \quad \text{and} \quad |P(\xi)| \leq \|P\|^{-\omega} \right\}.$$

Note that any element of $\mathcal{P}(\varepsilon)$ has norm at least $e^n H_0 > 1$. A classical argument of Wirsing [20, Hilfssatz 4] ensures that the set $\mathcal{P}(\varepsilon)$ is infinite (see also [9, Section 6]).

Lemma 4.1. *Let $\varepsilon \in (0,1)$ and let $P \in \mathcal{P}(\varepsilon)$. There are linearly independent polynomials $Q_1, \dots, Q_{n+1} \in \mathbb{Z}[X]_{\leq n}$ satisfying the following properties. Write $H_i = \|Q_i\|$ for $i = 1, \dots, n+1$.*

(i) The polynomials Q_1 and Q_2 are coprime and $Q_2 = P$.

(ii) We have $H_1 \leq \cdots \leq H_{n+1}$, and there exists $x \geq n$ such that $H_2 \cdots H_{n+1} = H_2^x$.

(iii) If $\|P\|$ is large enough, then $x \in [\widehat{\omega}_n(\xi) - \varepsilon, \omega_n(\xi) + \varepsilon]$ and

$$\max\{|Q_1(\xi)|, \dots, |Q_{n+1}(\xi)|\} \ll H_2^{-x+\varepsilon}. \tag{4.4}$$

The implicit constants depend on n and ξ only.

Proof. Recall from (4.1) that $\widehat{\omega} = \widehat{\omega}(\varepsilon)$ and $\omega = \omega(\varepsilon)$. Let $P \in \mathcal{P}(\varepsilon)$ and let $q \geq 0$ be maximal such that $L_1(q) = \log(e^{-n}\|P\|)$. The point q tends to infinity as $\|P\|$ goes to infinity. Let $Q \in \mathbb{Z}[X]_{\leq n}$ be such that

$$L(Q, q) = L_1(q).$$

By maximality of q, there exists $\eta > 0$ such that L_1 has slope 1 on $[q, q+\eta]$. Since $L_1 \leq L(Q, \cdot)$, the function $L(Q, \cdot)$ has slope 1 on $[q, \infty)$. Therefore

$$\log\|Q\| \leq L(Q, q) = \log|Q(\xi)| + q = L_1(q) = \log\left(e^{-n}\|P\|\right). \tag{4.5}$$

Thus $\|Q\| \leq e^{-n}\|P\|$, and, by (2.6), the irreducible polynomial P cannot be a factor of Q. They are therefore coprime. Moreover, Lemma 3.1 implies that if q (or equivalently $\|P\|$) is large enough, then Q is solution of (4.2), namely

$$\|Q\| \leq H \quad \text{and} \quad |Q(\xi)| \leq H^{-\widehat{\omega}},$$

with $H = e^{-n}\|P\| \geq H_0$. Combined with (4.3), this gives

$$\|Q\|^{-\omega_n(\xi)-\varepsilon/2} \leq |Q(\xi)| \leq \left(e^{-n}\|P\|\right)^{-\widehat{\omega}} \ll \|P\|^{-\widehat{\omega}}. \tag{4.6}$$

On the other hand, there exist $P_1, \dots, P_{n+1}$ in $\mathbb{Z}[X]_{\leq n}$, with $P_1 = Q$ satisfying assertions (i)–(iii) of Lemma 3.3. In particular, for $i = 2, \dots, n+1$, we have

$$\|P_i\| \geq e^{L_1(q)} = e^{-n}\|P\| \quad \text{and} \quad |P_i(\xi)| < |Q(\xi)|.$$

For these indices i, set $\widetilde{P}_i = P_i + \lambda_i P$ with $\lambda_i = 0$ if $\|P_i\| > \|P\|$, and $\lambda_i = 3$ otherwise, so that

$$\|P_i\| \asymp \|\widetilde{P}_i\| > \|P\| \quad \text{and} \quad |\widetilde{P}_i(\xi)| \leq 4\max\{|Q(\xi)|, |P(\xi)|\},$$

as well as

$$|P_1(\xi)| \prod_{i=2}^{n+1} \|\widetilde{P}_i\| \asymp |P_1(\xi)| \prod_{i=2}^{n+1} \|P_i\| \asymp 1. \tag{4.7}$$

Since the family $P, P_1, \widetilde{P}_2, \ldots, \widetilde{P}_{n+1}$ spans $\mathbb{R}[X]_{\leq n}$ and since P and $Q = P_1$ are linearly independent, there exists an index $j \in \{2, \ldots, n+1\}$ such that $P, P_1, \widetilde{P}_2, \ldots, \widehat{\widetilde{P}_j}, \ldots, \widetilde{P}_{n+1}$ are linearly independent (where $\widetilde{P}_j$ is omitted from the list). We denote by $Q_1, \ldots, Q_{n+1}$ this family reordered by increasing norm. By construction of the polynomials $\widetilde{P}_k$, we have $(Q_1, Q_2) = (Q, P)$. Let $x \in \mathbb{R}$ be such that

$$\|Q_2\| \cdots \|Q_{n+1}\| = \|Q_2\|^x.$$

The inequalities $\|Q_2\| \leq \cdots \leq \|Q_{n+1}\|$ imply that $x \geq n$. The first two assertions of the lemma are thus satisfied. We also have

$$\max\{|Q_1(\xi)|, \ldots, |Q_{n+1}(\xi)|\} \ll \max\{|Q(\xi)|, |P(\xi)|\}. \tag{4.8}$$

Since $\|P\| \leq \|\widetilde{P}_j\|$, we deduce from (4.7) that

$$1 \asymp |P_1(\xi)| \prod_{i=2}^{n+1} \|\widetilde{P}_i\| \geq |Q(\xi)| \prod_{i=2}^{n+1} \|Q_i\| = |Q(\xi)| \cdot \|P\|^x.$$

Therefore

$$|Q(\xi)| \ll \|P\|^{-x}. \tag{4.9}$$

According to the first inequality of (4.6) (and since $\|Q\| < \|P\|$), we have

$$\|P\|^x \ll \|P\|^{\omega_n(\xi)+\varepsilon/2}. \tag{4.10}$$

Consequently, as soon as $\|P\|$ is large enough, we have $x \in [n, \omega_n(\xi) + \varepsilon]$. It remains to prove the last assertion of our Lemma.

Proof of assertion (iii). Now, write $\mathcal{P}(\varepsilon)$ as a disjoint union

$$\mathcal{P}(\varepsilon) = \mathcal{P}_0(\varepsilon) \bigsqcup \mathcal{P}_1(\varepsilon),$$

where

$$\mathcal{P}_0(\varepsilon) = \{R \in \mathcal{P}(\varepsilon)\,;\, \log\left(e^{-n}\|R\|\right) < L_1(q_R)\} \quad \text{and} \quad \mathcal{P}_1(\varepsilon) = \mathcal{P}(\varepsilon) \setminus \mathcal{P}_0(\varepsilon),$$

and $q_R = \log\|R\| - \log|R(\xi)|$ as in Section 3. The set $\mathcal{P}_0(\varepsilon)$ is the set of polynomials $R \in \mathcal{P}(\varepsilon)$ which almost realize L_1 at $q = q_R$, since $L(R, q_R) < L_1(q_R) + n$. We only know that at least one of the sets $\mathcal{P}_0(\varepsilon), \mathcal{P}_1(\varepsilon)$ is infinite.

Case 1. First assume that $P \in \mathcal{P}_0(\varepsilon)$. Then $L_1(q_P) > \log(e^{-n}\|P\|)$ and we find

$$q < q_P = \log\|P\| - \log|P(\xi)|.$$

It follows that

$$\log\left(e^{-n}\|P\|\right) = L_1(q) = \log|Q(\xi)| + q \leq \log|Q(\xi)| + \log\|P\| - \log|P(\xi)|,$$

which implies $e^{-n}|P(\xi)| \le |Q(\xi)|$, and so (4.8) becomes

$$\max\{|Q_1(\xi)|, \dots, |Q_{n+1}(\xi)|\} \ll |Q(\xi)| \ll \|P\|^{-x},$$

hence (4.4). On the other hand, Lemma 3.2 applied to the family $(Q_1, \dots, Q_{n+1})$ ensures that

$$1 \ll |Q(\xi)| \prod_{i=2}^{n+1} \|Q_i\| = |Q(\xi)| \cdot \|P\|^{x}.$$

Combined with (4.9), this yields $|Q(\xi)| \asymp \|P\|^{-x}$. Then (4.6) gives $\|P\|^{\widehat{\omega}} \ll \|P\|^{x}$, and we conclude that $x \ge \widehat{\omega}_n(\xi) - \varepsilon$ as soon as $\|P\|$ is large enough.

Case 2. Assume that $P \in \mathcal{P}_1(\varepsilon)$. We now have $L_1(q_P) \le \log(e^{-n}\|P\|)$, and thus $q \ge q_P$. Combined with

$$e^{L(Q,q)} = e^{L_1(q)} = e^{-n}\|P\| < \|P\|$$

this implies $|Q(\xi)| = e^{L_1(q)-q} < \|P\| e^{-q_P} = |P(\xi)| \le \|P\|^{-\omega}$, and (4.8) becomes

$$\max\{|Q_1(\xi)|, \dots, |Q_{n+1}(\xi)|\} \ll |P(\xi)| \le \|P\|^{-\omega}. \tag{4.11}$$

Together with (4.10) this yields (4.4). Finally, Lemma 3.2 yields

$$1 \ll |P(\xi)| \prod_{i=2}^{n+1} \|Q_i\| \le \|P\|^{x-\omega},$$

and so $x \ge \omega - \varepsilon/2 = \omega_n(\xi) - \varepsilon \ge \widehat{\omega}_n(\xi) - \varepsilon$ as soon as $\|P\|$ is large enough, the last inequality coming from (2.3). □

5 Generalized resultants

Let n and ξ be as in Section 4. The main result of the section is the following, which, combined with Lemma 4.1, will allow us to construct algebraic numbers of degree at most n very close to ξ.

Proposition 5.1. *Let k be an integer with $2 \le k \le n+1$ and set $N = 2n - k + 1 \ge n$. Let $P_1, \dots, P_k \in \mathbb{Z}[X]_{\le n}$ be linearly independent polynomials, and write $H_i = \|P_i\|$ for $i = 1, \dots, k$. Suppose that P_1 and P_2 are coprime, and that*

$$H_1 \le \cdots \le H_k \quad \textit{and} \quad \max_{1 \le i \le k} |P_i(\xi)| \le \delta,$$

for some $\delta > 0$. Then, there exist an algebraic number α of degree $\le n$ and an index $m \in \{1, \dots, k\}$ such that

$$H(\alpha) \ll H_m \quad \textit{and} \quad |\xi - \alpha| \ll \delta^2 H_1^{n-k+1} H_2^{n-k+2} H_3 \cdots H_k H_m^{-1}, \tag{5.1}$$

where the implicit constants depend on n and ξ only.

To prove this result we will use generalized resultants. Let us recall the results from [12, §6]. We say that a function $g : \{n, n+1, n+2 \cdots\} \to \mathbb{R}$ is *concave* if

$$g(i) - g(i-1) \geq g(i+1) - g(i)$$

for any $i > n$. Let $N \geq n$ be an integer and let $\mathcal{A} \neq \{0\}$ be a subset of $\mathbb{R}[X]_{\leq n}$ containing a non-zero element. We define

$$\begin{aligned}
&\mathcal{B}_N(\mathcal{A}) = \{Q, XQ, \dots, X^{N-\deg(Q)}Q\,;\, Q \in \mathcal{A} \setminus \{0\}\} \subseteq \mathbb{R}[X]_{\leq N},\\
&V_N(\mathcal{A}) = \langle \mathcal{B}_N(\mathcal{A}) \rangle_{\mathbb{R}}\,,\\
&g_{\mathcal{A}}(N) = \dim V_N(\mathcal{A}).
\end{aligned}$$

We call *generalized resultant* the determinant of any $N+1$ elements chosen in $\mathcal{B}_N(\mathcal{A})$, for some $\mathcal{A}$ as above. According to [12, Lemma 6.3], the function $g_{\mathcal{A}}$ is (strictly) increasing and concave on $\{n, n+1, \dots\}$. If we assume furthermore that the gcd of the elements of $\mathcal{A}$ is 1 (in other words the ideal spanned by $\mathcal{A}$ is $\mathbb{R}[X]$), then

$$V_{2n-1}(\mathcal{A}) = \mathbb{R}[X]_{\leq 2n-1} \tag{5.2}$$

(it is a direct consequence of [12, Proposition 6.2]).

Lemma 5.2. *Let $\mathcal{A}$ be a linearly independent subset of $\mathbb{R}[X]_{\leq n}$ of cardinality j with $2 \leq j \leq n+1$. We also suppose that the* gcd *of the elements of $\mathcal{A}$ is* 1*. Then, for $h = 0, \dots, n-j+1$, we have*

$$\dim V_{n+h}(\mathcal{A}) \geq 2h + j.$$

Proof. By contradiction, suppose that there exists $h \in \{0, \dots, n-j+1\}$ such that

$$g_{\mathcal{A}}(n+h) < 2h + j.$$

Since $g_{\mathcal{A}}(n) \geq \operatorname{card}(\mathcal{A}) = j$, we have $h \geq 1$.

Case 1. Suppose that $g_{\mathcal{A}}(n+h) \geq g(n+h-1) + 2$. By concavity, we have $g_{\mathcal{A}}(i) \geq g_{\mathcal{A}}(i-1) + 2$ for $i = n+1, \dots, n+h$, and we deduce that

$$g_{\mathcal{A}}(n+h) \geq 2h + g_{\mathcal{A}}(n) \geq 2h + j$$

which is a contradiction.

Case 2. So $g_{\mathcal{A}}(n+h) \leq g(n+h-1) + 1$. By concavity (and since $g_{\mathcal{A}}$ is increasing), we have $g_{\mathcal{A}}(i+1) = g_{\mathcal{A}}(i) + 1$ for $i = n+h, \dots, 2n$. Combined with (5.2), we get

$$2n = g_{\mathcal{A}}(2n-1) = g_{\mathcal{A}}(n+h) + 2n - 1 - (n+h) < n + h + j - 1 \leq 2n$$

(the last inequality coming from $h \leq n - j + 1$), which is, once again, a contradiction. □

As a corollary, we obtain the following useful result.

Corollary 5.3. *Let k, N be as in Proposition 5.1. Let $P_1, \dots, P_k \in \mathbb{Z}[X]_{\le n}$ be linearly independent polynomials such that P_1 and P_2 are coprime. Then, for each $j = 2, \dots, k$ we have*

$$\dim V_N(P_1, \dots, P_j) \ge 2(n-k+1) + j.$$

In particular,

$$V_N(P_1, \dots, P_k) = \mathbb{R}[X]_{\le N}.$$

Proof. Set $\mathcal{A} = \{P_1, \dots, P_k\}$. Fix an integer j with $2 \le j \le k$ and choose $h = n - k + 1 = N - n$. Then $0 \le h \le n - j + 1$, and Lemma 5.2 yields

$$\dim V_N(P_1, \dots, P_k) = \dim V_{n+h}(\mathcal{A}) \ge 2h + j = 2(n-k+1) + j.$$

□

Proof of Proposition 5.1. First, note that there exist $\lambda_1, \lambda_2 \in \{0, \dots, n\}$ such that the polynomials

$$Q_i = (X - \lambda_i)^{n - \deg(P_i)} P_i \qquad (i = 1, 2)$$

are coprime and of degree exactly n. By Gel'fond's Lemma, they also satisfy $\|Q_i\| \asymp \|P_i\| = H_i$ and $|Q_i(\xi)| \asymp |P_i(\xi)| \le \delta$ $(i = 1, 2)$, and the vector space

$$F = V_N(Q_1, Q_2)$$

spanned by $Q_1, XQ_1, \dots, X^{n-k+1}Q_1, Q_2, XQ_2, \dots, X^{n-k+1}Q_2$ has dimension $2(n-k+2)$. We can choose a subsequence $(Q_3, \dots, Q_k)$ of $(P_1, \dots, P_k)$ such that $Q_1, \dots, Q_k$ are linearly independent. For each $j = 3, \dots, k$ there is some $i \in \{1, \dots, j\}$ such that $\|Q_j\| = H_i \le H_j$. According to Corollary 5.3, we have

$$\dim \big(F + V_N(Q_3, \dots, Q_j)\big) = \dim V_N(Q_1, \dots, Q_j) \ge \dim F + j - 2,$$

for $j = 2, \dots, k$. For $j = k$ we obtain $V_N(Q_1, \dots, Q_k) = \mathbb{R}[X]_{\le N}$. By recurrence, for $j = 3, \dots, k$, we choose $R_j \in \mathcal{B}_N(Q_3, \dots, Q_j)$ such that

$$\dim \big(F + \langle R_3, \dots, R_j \rangle_{\mathbb{R}}\big) = \dim F + j - 2.$$

In particular

$$F \oplus \langle R_3, \dots, R_k \rangle_{\mathbb{R}} = \mathbb{R}[X]_{\le N}. \tag{5.3}$$

Note that for each $j = 3, \dots, k$, there is some index $i \in \{1, \dots, j\}$ such that,

$$\|R_j\| = H_i \le H_j \quad \text{and} \quad |R_j(\xi)| \ll \delta. \tag{5.4}$$

Moreover, the roots of R_j are algebraic numbers of degree at most n, since they are either 0 or a root of one of the polynomials $Q_3, \dots, Q_j \in \mathbb{Z}[X]_{\le n}$. By (5.3), the sequence

$$(S_0, \dots, S_N) = (Q_1, XQ_1, \cdots, X^{n-k+1}Q_1, Q_2, XQ_2, \cdots, X^{n-k+1}Q_2, R_3, \dots, R_k)$$

forms a basis of $\mathbb{R}[X]_{\leq N}$. The first $n-k+2$ polynomials S_i have norm $\asymp H_1$, while the following $n-k+2$ ones have norm $\asymp H_2$. We control the norms of the last $k-2$ polynomials $R_3,\dots,R_k$ using (5.4). The corresponding non-zero generalized resultant $\det(S_0,\dots,S_N)$ satisfies

$$1 \leq |\det(S_0,\dots,S_N)| = \left|\det\left(S_j^{[i]}(\xi)\right)_{0\leq i,j\leq N}\right|,$$

see (2.2) for the last equality. For $j=0,\dots,N$, we have

$$|S_j^{[0]}(\xi)| = |S_j(\xi)| \ll \delta \quad \text{and} \quad S_j^{[1]}(\xi) = S_j'(\xi).$$

For $i=2,\dots,N$ we will use the crude estimate $|S_j^{[i]}(\xi)| \ll \|S_j\|$. Expanding the last determinant, we obtain

$$1 \leq \left|\det\left(S_j^{[i]}(\xi)\right)_{0\leq i,j\leq N}\right| \ll \delta H_1^{n-k+1} H_2^{n-k+2} H_3\cdots H_k \sum_{\ell=0}^{N} \frac{|S_\ell'(\xi)|}{\|S_\ell\|}. \tag{5.5}$$

Let $\ell\in\{0,\dots,N\}$ be such that $|S_\ell'(\xi)|/\|S_\ell\|^{-1}$ is maximal, and let α be a root of S_ℓ such that $|\xi-\alpha|$ is minimal. Recall that α is algebraic of degree at most n and that there exists $m\in\{1,\dots,k\}$ such that $\|S_\ell\|\asymp H_m$. Then, the minimal polynomial of α divides S_ℓ, and Gel'fond's lemma yields $H(\alpha)\ll\|S_\ell\|\ll H_m$. On the other hand, by (2.5) we have

$$|\xi-\alpha||S_\ell'(\xi)| \ll |S_\ell(\xi)| \ll \delta.$$

Multiplying both sides of (5.5) by $|\xi-\alpha|$, this yields (5.1). □

Proposition 5.1 has the following Corollary.

Corollary 5.4. *Let k be an integer with $2\leq k\leq n+1$ and let $C,y>0$. Let $P_1,\dots,P_k\in\mathbb{Z}[X]_{\leq n}$ be linearly independent polynomials and write $H_i=\|P_i\|$ for $i=1,\dots,k$. Assume that*

(i) P_1 and P_2 are coprime and $H_2\geq 2$;

(ii) $H_1\leq\cdots\leq H_k$;

(iii) $|P_i(\xi)|\leq CH_2^{-y}$ for $i=1,\dots,k$.

For $i=2,\dots,k$ write $H_i=H_2^{a_i}$, and suppose furthermore that

$$A_k := 2y-2(n+1-k)-a_2-\cdots-a_k \geq 0.$$

Then, there exist an algebraic number α of degree $\leq n$ and a constant c which depends on n, ξ only, such that

$$|\xi-\alpha| \ll C^2\min\left\{\left(cH(\alpha)\right)^{-A_k/a_k-1}, H_2^{-A_k-1}\right\}. \tag{5.6}$$

The implicit constant depends on n and ξ only.

Remark 5.5. Since $A_k \geq 0$, equation (5.6) implies that $|\xi - \alpha| \ll 1/H_2$ tends to 0 as H_2 tends to infinity. Consequently $H(\alpha)$ tends to infinity as H_2 tends to infinity.

Proof. Set $\delta = CH_2^{-y}$. By Proposition 5.1, there exist an algebraic number α of degree at most n and $m \in \{2, \dots, k\}$ such that

$$cH(\alpha) \leq H_m \quad \text{and} \quad |\xi - \alpha| \ll \delta^2 H_2^{2n-2k+3} H_3 \cdots H_k H_m^{-1} = C^2 H_2^{-A_k - a_m}, \tag{5.7}$$

where $c > 0$ depends on ξ and n only. Since $a_m \geq 1$, we have $|\xi - \alpha| \ll C^2 H_2^{-A_k - 1}$. Furthermore, using $a_k \geq a_m$ and $A_k \geq 0$, Estimates (5.7) also yield

$$|\xi - \alpha| \ll C^2 H_m^{-A_k/a_m - 1} \leq C^2 H_m^{-A_k/a_k - 1} \leq C^2 (cH(\alpha))^{-A_k/a_k - 1}.$$

□

6 A step toward Wirsing's conjecture

Let n and ξ be as in Section 4. In this section, we merge the main results of the preceding two sections to provide a lower bound for $\omega_n^*(\xi)$. This uses the following notation. Given $x \geq n$ we define

$$\mathcal{A}(x) = \Big\{ \mathbf{a} = (a_2, \dots, a_{n+1}) \in \mathbb{R}^n \,;\, 1 = a_2 \leq \cdots \leq a_{n+1} \quad \text{and} \quad a_2 + \cdots + a_{n+1} = x \Big\}.$$

For each $\mathbf{a} = (a_2, \dots, a_{n+1}) \in \mathcal{A}(x)$ and each integer k with $2 \leq k \leq n+1$, we set

$$A_k(x, \mathbf{a}) = 2x - 2(n - k + 1) - \sum_{i=2}^{k} a_i = 2(x - n) + \sum_{i=2}^{k} (2 - a_i),$$

and

$$F(x, \mathbf{a}) = \max_{2 \leq k \leq n+1} \frac{A_k(x, \mathbf{a})}{a_k}.$$

Since $\mathbf{a} \mapsto F(x, \mathbf{a})$ is continuous on the compact set $\mathcal{A}(x)$, we may also define

$$F(x) = \min_{\mathbf{a} \in \mathcal{A}(x)} F(x, \mathbf{a}).$$

Note that the condition $a_2 + \cdots + a_{n+1} = x$ in the definition of $\mathcal{A}(x)$ is equivalent to $x = A_{n+1}(x, \mathbf{a})$. Furthermore, for each $\mathbf{a} \in \mathcal{A}(x)$, we have

$$A_k(x, \mathbf{a}) \geq 1 \qquad \text{for } k = 2, \dots, n+1, \tag{6.1}$$

since $2(n - k + 1) + a_2 + \cdots + a_k \leq (n - k + 1) + a_2 + \cdots + a_{n+1} \leq 2x - 1$.

We claim that the function $x \mapsto F(x)$ is continuous on $[n, +\infty)$. Indeed, let M, x, y be real numbers with $M \geq y \geq x \geq n$ and write $\delta = y - x$. Choose $\mathbf{a} \in \mathcal{A}(x)$ such that $F(x) = F(x, \mathbf{a})$, and denote by $\mathbf{b}'$ the point obtained by adding δ to the last coordinate of $\mathbf{a}$. Then $\mathbf{b}' \in \mathcal{A}(y)$ and

$$\|\mathbf{a} - \mathbf{b}'\| \leq \delta.$$

This implies that

$$F(x) = F(x, \mathbf{a}) = F(y, \mathbf{b}') + \mathcal{O}(\delta) \geq F(y) + \mathcal{O}(\delta),$$

where the implicit constant depend on M only. Similarly, we show that $F(y) \geq F(x) + \mathcal{O}(\delta)$ by choosing $\mathbf{b} \in \mathcal{A}(y)$ satisfying $F(y) = F(y, \mathbf{b})$ and a point $\mathbf{a}' \in \mathcal{A}(x)$ with $\|\mathbf{b} - \mathbf{a}'\| \leq \delta$ (whose existence we leave to the reader). Thus, $|F(y) - F(x)| = \mathcal{O}(|y - x|)$ for any $x, y \in [n, M]$, and our claim follows.

Theorem 6.1. *We have*

$$\omega_n^*(\xi) \geq \inf\{F(x)\,;\, \omega_n(\xi) \geq x \geq \widehat{\omega}_n(\xi)\}.$$

Consequently,

$$\omega_n^*(\xi) \geq F_n := \inf_{x \geq n} F(x).$$

Proof. Fix a small $\varepsilon \in (0, 1/2)$ and let P be an element of the infinite set $\mathcal{P}(\varepsilon)$ defined as in Section 4. According to Lemma 4.1, if $\|P\|$ is large enough, then there exist linearly independent polynomials $Q_1, \ldots, Q_{n+1} \in \mathbb{Z}[X]_{\leq n}$ and $x \geq n$ with

$$\widehat{\omega}_n(\xi) - \varepsilon \leq x \leq \omega_n(\xi) + \varepsilon,$$

such that, writing $H_i = \|Q_i\|$ for $i = 1, \ldots, n+1$, we have

(i) Q_1 and Q_2 are coprime, with $Q_2 = P$;

(ii) $H_1 \leq \cdots \leq H_{n+1}$ and $H_2 \cdots H_{n+1} = H_2^x$;

(iii) $|Q_1(\xi)|, \ldots, |Q_{n+1}(\xi)| \ll H_2^{-x+\varepsilon}$.

For $i = 2, \ldots, n+1$, define $a_i \geq 1$ by $H_i = H_2^{a_i}$. Condition (ii) means that the point $\mathbf{a} = (a_2, \ldots, a_{n+1})$ belongs to $\mathcal{A}(x)$. Set $y = x - \varepsilon$. By (6.1), for each $k \in \{2, \ldots, n+1\}$, we have

$$2y - 2(n + 1 - k) - a_2 - \cdots - a_k = A_k(x, \mathbf{a}) - 2\varepsilon \geq 0.$$

By Corollary 5.4 applied successively with $k = 2, \ldots, n+1$, there exists an algebraic number α of degree at most n, such that

$$|\xi - \alpha| \ll H(\alpha)^{-F(x,\mathbf{a})-1+2\varepsilon} \leq H(\alpha)^{-F(x)-1+2\varepsilon} \leq H(\alpha)^{-F_n(\varepsilon)-1+2\varepsilon},$$

where $F_n(\varepsilon)$ denotes the minimum of F on $[\widehat{\omega}_n(\xi) - \varepsilon, \omega_n(\xi) + \varepsilon] \cap [n, +\infty)$. Recall that $H(\alpha)$ tends to infinity with $\|P\|$, because we have $|\xi - \alpha| \ll H_2^{-1} = \|P\|^{-1}$ in view of the last estimate of Corollary 5.4. Since $\mathcal{P}(\varepsilon)$ is infinite, we deduce that

$$\omega_n^*(\xi) \geq F_n(\varepsilon) - 2\varepsilon.$$

Since F is continuous on $[n, +\infty)$, we get the result by letting ε tend to 0. □

Remark 6.1. Suppose that in the proof of Lemma 4.1, the set $\mathcal{P}_1(\varepsilon)$ is infinite for arbitrarily small values of ε. Then we could take $x \geq \omega_n(\xi) - \varepsilon$ in the proof of Theorem 6.1, and we would obtain $\omega_n^*(\xi) \geq F(\omega_n(\xi))$.

7 A minimization problem

Let the notation be as in Section 6. Theorem 6.1 calls for a lower bound estimate for F_n. We first prove that there exist a point $(x, \mathbf{a}) \in \mathbb{R}^{n+1}$ with $x \geq n$ and $\mathbf{a} \in \mathcal{A}(x)$ satisfying $F(x, \mathbf{a}) = F_n$. Then, we give a complete description of $\mathbf{a}$ as a function of F_n and x and some integer ℓ with $2 \leq \ell \leq n$. In the final Section 8, we use these properties to give an explicit lower bound for F_n and deduce Theorem 1.1. Our approach is inspired by the remarkable strategy described by de La Vallée-Poussin in [10, Chapter VI] to construct polynomials of best approximation to a continuous real valued function on a closed interval on $\mathbb{R}$.

Theorem 7.1. *There exists a point $(x, \mathbf{a}) \in \mathbb{R}^{n+1}$, with $\mathbf{a} = (a_2, \dots, a_{n+1})$, such that*

$$x \geq n, \quad \mathbf{a} \in \mathcal{A}(x) \quad \text{and} \quad F_n = F(x, \mathbf{a}). \tag{7.1}$$

Any such point has the following properties.

(i) There exists $\ell \in \{2, \dots, n\}$ such that $F_n = 2(x - n) + \ell - 1$ and

$$x = (2 - \theta) F_n, \quad \text{where } \theta = \left(\frac{F_n}{F_n + 1} \right)^{n+1-\ell}.$$

(ii) The point $\mathbf{a} = (a_2, \dots, a_{n+1})$ is given by $a_2 = \cdots = a_\ell = 1$, and

$$a_k = 2 - \left(\frac{F_n}{F_n + 1} \right)^{k-\ell}, \qquad \text{for } k = \ell, \dots, n+1.$$

(iii) We have

$$F_n = \frac{A_{\ell+1}(x, \mathbf{a})}{a_{\ell+1}} = \cdots = \frac{A_{n+1}(x, \mathbf{a})}{a_{n+1}}.$$

Theorem 7.1 implies that there are at most $n - 1$ points satisfying (7.1) (for such a point is entirely determined by the integer ℓ). Note that the first part of (i) combined with $a_\ell = 1$ ensures that the formula in (iii) is also valid for the index ℓ. In order to prove the above theorem, we first prove that the infimum F_n is actually a minimum.

Lemma 7.1. *We have $F_n < n$, and the set $\mathcal{M}_n$ of points $(x, \mathbf{a}) \in \mathbb{R}^{n+1}$ satisfying (7.1) is non-empty. Furthermore, any $(x, \mathbf{a}) \in \mathcal{M}_n$ has $n < x < (3n-1)/2$.*

Proof. For a fixed $\varepsilon \in [0, 1/2)$, the point $\mathbf{a} = (1, \dots, 1, 1 + \varepsilon) \in \mathbb{R}^n$ belongs to $\mathcal{A}(x)$ with $x = n + \varepsilon$. It follows from the definition that $A_k(x, \mathbf{a})/a_k = 2\varepsilon + k - 1 < n$ for $k = 2, \dots, n$ and

$$\frac{A_{n+1}(x, \mathbf{a})}{a_{n+1}} = \frac{n + \varepsilon}{1 + \varepsilon} \leq n,$$

with equality if and only if $\varepsilon = 0$. Taking $0 < \varepsilon < 1/2$, we deduce that $F_n \leq F(x, \mathbf{a}) < n$. Note that for $x = n$, the set $\mathcal{A}(n)$ reduces to $\{(1, \dots, 1)\}$ and $F(n) = n$. On the other hand, for any $x \geq n$, each $\mathbf{a} = (a_2, \dots, a_{n+1}) \in \mathcal{A}(x)$ has $a_2 = 1$, thus

$$F(x, \mathbf{a}) \geq \frac{A_2(x, \mathbf{a})}{a_2} = 2(x - n) + 1.$$

If follows that $F(x) \geq 2(x-n)+1$. Consequently, if $x \geq (3n-1)/2$, then $F(x) \geq n > F_n$. Consider the compact subset $\mathcal{K}_n$ of $\mathbb{R}^{n+1}$ given by

$$\mathcal{K}_n = \left\{ (x, \mathbf{a}) \in \mathbb{R}^{n+1} \mid x \in \left[n, \frac{3n-1}{2} \right] \quad \text{and} \quad \mathbf{a} \in \mathcal{A}(x) \right\}.$$

By the above, we have $F_n = \inf_{(x,\mathbf{a}) \in \mathcal{K}_n} F(x, \mathbf{a})$. Since the function F is continuous on the compact set $\mathcal{K}_n$, this infimum is actually a minimum. Furthermore, since $F_n < n$, any point $(x, \mathbf{a}) \in \mathcal{K}_n$ realizing this minimum satisfies $n < x < (3n-1)/2$. □

Lemma 7.2. *Let $\mathcal{M}_n$ be as in Lemma 7.1, let $(x, \mathbf{a}) \in \mathcal{M}_n$ and write $\mathbf{a} = (a_2, \ldots, a_{n+1})$. There exists an integer $\ell \in \{2, \ldots, n\}$ such that*

(i) $1 = a_2 = \cdots = a_\ell < a_{\ell+1} < \cdots < a_{n+1} < 2$;

(ii) $A_{\ell+1}(x, \mathbf{a})/a_{\ell+1} = \cdots = A_{n+1}(x, \mathbf{a})/a_{n+1} = F_n$;

(iii) $2(x-n) + \ell - 1 \leq F_n < 2(x-n) + \ell$.

Proof. **Step 1.** Suppose that $a_j < a_{j+1}$ for an integer j with $2 \leq j \leq n$. We claim that

$$\frac{A_{j+1}(x, \mathbf{a})}{a_{j+1}} = F_n. \tag{7.2}$$

Indeed, for each $\varepsilon \in (0, a_{j+1} - a_j]$, the point

$$\mathbf{b} = (b_2, \ldots, b_{n+1}) = (a_2, \ldots, a_j, a_{j+1} - \varepsilon, a_{j+2}, \ldots, a_{n+1})$$

belongs to $\mathcal{A}(y)$, where $y = x - \varepsilon$. Since $b_2 = 1$, we have $y = b_2 + \cdots + b_{n+1} \geq n$. By definition of the functions A_k we have

$$A_k(y, \mathbf{b}) = \begin{cases} A_k(x, \mathbf{a}) - 2\varepsilon & \text{for } k = 2, \ldots, j, \\ A_k(x, \mathbf{a}) - \varepsilon & \text{for } k = j+1, \ldots, n+1. \end{cases}$$

So, for each $k \neq j+1$, we find

$$\frac{A_k(y, \mathbf{b})}{b_k} < \frac{A_k(x, \mathbf{a})}{a_k} \leq F(x, \mathbf{a}) = F_n.$$

However, by minimality of F_n, we have $F(y, \mathbf{b}) \geq F_n$, thus

$$F(y, \mathbf{b}) = \frac{A_{j+1}(y, \mathbf{b})}{b_{j+1}} = \frac{A_{j+1}(x, \mathbf{a}) - \varepsilon}{a_{j+1} - \varepsilon} \geq F_n.$$

Letting ε tend to 0, we obtain $F_n \leq A_{j+1}(x, \mathbf{a})/a_{j+1} \leq F(x, \mathbf{a})$, hence our claim.

Step 2. Suppose that $a_{n+1} \geq 2$. Since $a_2 = 1 < 2$, there exists an integer j with $2 \leq j \leq n$ such that $a_j < 2 \leq a_{j+1}$. Using Step 1, we get

$$F_n = F(x, \mathbf{a}) \geq \frac{A_j(x, \mathbf{a})}{a_j} = \frac{A_{j+1}(x, \mathbf{a}) + a_{j+1} - 2}{a_j} > \frac{A_{j+1}(x, \mathbf{a})}{a_{j+1}} = F_n,$$

a contradiction. Hence $1 = a_2 \leq a_3 \leq \cdots \leq a_{n+1} < 2$.

Step 3. Let ℓ be the largest integer in $\{2, \ldots, n+1\}$ such that $a_\ell = 1$. As $1 = a_2 \leq \cdots \leq a_{n+1}$, we have $a_2 = \cdots = a_\ell = 1$. If $\ell = n+1$, then $x = n$, which contradicts Lemma 7.1, so $\ell \leq n$. By contradiction, suppose that assertion (i) is false. Then $\ell < n$ and by Step 2 there exists an integer j with $\ell \leq j \leq n-1$ such that $a_j < a_{j+1} = a_{j+2} < 2$. Using Step 1, we obtain

$$F_n = F(x, \mathbf{a}) \geq \frac{A_{j+2}(x, \mathbf{a})}{a_{j+2}} = \frac{A_{j+1}(x, \mathbf{a}) + 2 - a_{j+2}}{a_{j+1}} > \frac{A_{j+1}(x, \mathbf{a})}{a_{j+1}} = F_n,$$

a contradiction. Thus (i) holds, and by Step 1, it yields (ii). Finally, assertion (iii) follows from $a_{\ell+1} > 1$ and

$$2(x-n) + \ell - 1 = \frac{A_\ell(x, \mathbf{a})}{a_\ell} \leq F_n = \frac{A_{\ell+1}(x, \mathbf{a})}{a_{\ell+1}} < A_{\ell+1}(x, \mathbf{a}) < 2(x-n) + \ell.$$

□

Lemma 7.3. *Let $x \in [n, (3n-1)/2]$, let $\ell \in \{2, \ldots, n\}$, let $\mathbf{a} = (a_2, \ldots, a_{n+1}) \in \mathbb{R}^n$ with $a_2 = \cdots = a_\ell = 1$ and let $y, F \in \mathbb{R}$ with*

$$y = 2(x-n) + \ell - 1 \leq F < y + 1.$$

The following assertions are equivalent

(i) For $k = \ell+1, \ldots, n+1$, we have

$$\frac{A_k(x, \mathbf{a})}{a_k} = F. \tag{7.3}$$

(ii) For $k = \ell+1, \ldots, n+1$, we have

$$a_k = 2 - \frac{2F - y}{F+1} \left(\frac{F}{F+1} \right)^{k-\ell-1}. \tag{7.4}$$

If they hold, then $1 < a_{\ell+1} < \cdots < a_{n+1} < 2$.

Proof. (i) ⇔ (ii). As $A_{\ell+1}(x, \mathbf{a}) = y + 2 - a_{\ell+1}$, we first observe that (7.3) holds for $k = \ell+1$ if and only if $a_{\ell+1} = (y+2)/(F+1)$. Suppose that (7.3) holds for an index k with $\ell+1 \leq k \leq n$. Then, since $A_{k+1}(x, \mathbf{a}) = A_k(x, \mathbf{a}) + 2 - a_{k+1}$, the equality (7.3) holds for $k+1$ if and only if

$$a_{k+1} = \frac{F}{F+1} a_k + \frac{2}{F+1}. \tag{7.5}$$

By the above remark, (i) holds if and only if $a_{\ell+1} = (y+2)/(F+1)$ and (7.5) is satisfied for $k = \ell+1, \ldots, n$. This is precisely the arithmetico-geometric sequence of (ii).

Finally, the hypothesis $y \leq F < y+1$ implies that

$$\frac{F}{F+1} \leq \frac{2F - y}{F+1} < 1,$$

so that if (7.4) holds, then $a_2 = \cdots = a_\ell < a_{\ell+1} < \cdots < a_{n+1}$. □

Lemma 7.4. *With the notation and hypotheses of Lemma 7.3 set*

$$\theta = \left(\frac{F}{F+1}\right)^{n+1-\ell},$$

and suppose that $\mathbf{a} \in \mathbb{R}^n$ *satisfies the two equivalent conditions (i) and (ii) of Lemma 7.3, as well as* $1 = a_2 = \cdots = a_\ell$. *Then,* $(x, \mathbf{a}) \in \mathcal{A}(x)$ *if and only if*

$$(1-2\theta)x = 2F(1-\theta) - \theta(2n-\ell+1). \tag{7.6}$$

Moreover, if (7.6) *holds, then* $\theta \neq 1/2$.

Proof. Since $y = 2(x-n)+\ell-1$, Equation (7.6) is equivalent to $x = 2F - (2F-y)\theta$. Recall that $a_2 + \cdots + a_{n+1} = x$ if and only if $x = A_{n+1}(\mathbf{a}, x)$. Since the coordinates of $\mathbf{a}$ are increasing, the point $(x, \mathbf{a})$ belongs to $\mathcal{A}(x)$ if, and only if,

$$x = A_{n+1}(x, \mathbf{a}) \underset{(7.3)}{=} a_{n+1}F \underset{(7.4)}{=} 2F - \frac{F(2F-y)}{F+1}\left(\frac{F}{F+1}\right)^{n-\ell} = 2F - (2F-y)\theta.$$

It remains to prove the last part of the Lemma. By contradiction, suppose now that (7.6) holds with $\theta = 1/2$. Then, we obtain

$$F = \frac{2n-\ell+1}{2} \in \mathbb{Q} \quad \text{and} \quad \frac{1}{2} = \theta = \left(\frac{F}{F+1}\right)^{n+1-\ell}.$$

This implies that the exponent $n+1-\ell$ is equal to 1, thus $\ell = n$ and $F = (n+1)/2 = 1$, which is impossible since $n \geq 2$. □

Proof of Theorem 7.1. By Lemma 7.1 there exists $(x, \mathbf{a}) = (x, a_2, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ satisfying (7.1), and any such point has $n < x < (3n-1)/2$. Fix such a point. Then Lemma 7.2 provides an integer $\ell \in \{2, \ldots, n\}$ for which $a_2 = \cdots = a_\ell = 1$,

$$y := 2(x-n)+\ell-1 \leq F_n < y+1, \tag{7.7}$$

and assertion (iii) of Theorem 7.1 holds. It only remains to prove that $F_n = y$ and that $x = (2-\theta)F_n$, for then Lemma 7.3 implies assertions (i) and (ii) of the theorem. According to Lemma 7.4, we have

$$(1-2\theta)x = 2F_n(1-\theta) - \theta(2n-\ell+1), \quad \text{where } \theta = \left(\frac{F_n}{F_n+1}\right)^{n+1-\ell} \neq \frac{1}{2}.$$

Fix $\varepsilon \in [0,1)$ and set $F' = F_n - \varepsilon$. If ε is small enough, then

$$\theta' := \left(\frac{F'}{F'+1}\right)^{n+1-\ell} \neq \frac{1}{2},$$

and there exists $x' = x'(\varepsilon) \in \mathbb{R}$ such that (x', θ', F') satisfy (7.6). By contradiction, suppose that $y < F_n < y+1$. We note that for $\varepsilon = 0$, we have $(x', F') = (x, F_n)$. So, if ε is small enough, we also have $n < x' < (3n-1)/2$ and $y' < F' < y'+1$, where

$$y' = y'(\varepsilon) = 2(x'-n)+\ell-1.$$

Set $a'_2 = \cdots = a'_\ell = 1$ and define a'_k by (7.4) (with $F = F'$) for $k = \ell+1, \dots, n+1$. We denote by $\mathbf{a}'$ the point $(a'_2, \dots, a'_{n+1})$. Then x', $\mathbf{a}'$, ℓ, y' and F' satisfy the hypotheses of Lemmas 7.3 and 7.4. According to Lemma 7.4, and since x', F' satisfy (7.6), we have $\mathbf{a}' \in \mathcal{A}(x')$. Moreover, $A_k(x', \mathbf{a}')/a'_k = 2(x'-n) + k - 1 \le y'$ for $k = 2, \dots, \ell$, and our choice of $\mathbf{a}'$ yields

$$\frac{A_k(x', \mathbf{a}')}{a'_k} = F' \ge y' \qquad \text{for } k = \ell+1, \dots, n+1.$$

Thus $F(x', \mathbf{a}') = F' < F_n$, a contradiction. It follows that $F_n = y$, as expected, hence assertion (ii) of Theorem 7.1 holds. In particular, the last coordinate of $\mathbf{a}$ multiplied by F_n is equal to $(2-\theta)F_n$ by assertion (ii), and is also equal to $A_{n+1}(x, \mathbf{a}) = x$ by assertion (iii). Hence the identity $(2-\theta)F_n = x$. □

8 Proof of the main result

Let n be an integer ≥ 2. We keep the notation of Section 6 for the function F and its minimum F_n. We now have all the tools we need to give an explicit lower bound for F_n. Together with Theorem 6.1, the next estimate implies Theorem 1.1.

Theorem 8.1. *We have $F_n \ge n/(2 - \log 2)$.*

Proof. Fix $(x, \mathbf{a}) \in \mathbb{R}^{n+1}$ satisfying the condition (7.1) of Theorem 7.1, and let $\ell \in \{2, \dots, n\}$ such that

$$F_n = 2(x-n) + \ell - 1. \tag{8.1}$$

Set $\theta = \big(F_n/(F_n+1)\big)^{n+1-\ell}$. The formula $x = (2-\theta)F_n$ combined with (8.1) leads to

$$(3 - 2\theta)F_n = 2n + 1 - \ell. \tag{8.2}$$

Since $t \log(1 + 1/t) \le 1$ for each $t > 0$, we find

$$F_n \log\theta = -(n+1-\ell)F_n \log\left(1 + \frac{1}{F_n}\right) \ge -(n+1-\ell).$$

Together with (8.2), this yields $(3 - 2\theta + \log\theta)F_n \ge n$. Finally, the function $t \mapsto 3 - 2t + \log t$ has a global maximum on $(0, \infty)$ at $t = 1/2$, which is equal to $2 - \log 2$, hence $(2 - \log 2)F_n \ge n$. □

Acknowledgements: I am very grateful to Damien Roy for his encouragement and the many fruitful discussions we had. His careful reading of this work and his many pertinent suggestions helped improve the overall presentation and simplify some arguments. I also thank the referees for their comments and suggestions.

References

[1] D. Badziahin and J. Schleischitz. An improved bound in Wirsing's problem. *Trans. Amer. Math. Soc.*, 374(3):1847–1861, 2021.

[2] V. I. Bernik and K. Tishchenko. Integral polynomials with an overfall of the coefficient values and Wirsing's theorem. In *Dokl. Akad. Nauk Belarusi*, volume 37, pages 9–11, 1993.

[3] É. Borel. Contribution à l'analyse arithmétique du continu. *J. Math. Pures Appl.*, 9:329–375, 1903.

[4] W. D. Brownawell. Sequences of Diophantine approximations. *J. Number Theory*, 6(1):11–21, 1974.

[5] Y. Bugeaud. *Approximation by algebraic numbers*, volume 160 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 2004.

[6] Y. Bugeaud. Exponents of Diophantine approximation. In D. Badziahin, A. Gorodnik, and N. Peyerimhoff, editors, *Dynamics and Analytic Number Theory*, volume 437 of *London Math. Soc. Lecture Note Ser.*, chapter 2, pages 96–135. Cambridge University Press, 2016.

[7] H. Davenport and W. Schmidt. Approximation to real numbers by quadratic irrationals. *Acta Arith.*, 13(2):169–176, 1967.

[8] H. Davenport and W. Schmidt. A theorem on linear forms. *Acta Arith.*, 2(14):209–223, 1968.

[9] H. Davenport and W. Schmidt. Approximation to real numbers by algebraic integers. *Acta Arith.*, 15(4):393–416, 1969.

[10] C. De La Vallée Poussin. *Leçons sur l'approximation des fonctions d'une variable réelle.* Gauthier-Villars, Paris, 1919.

[11] J. F. Koksma. Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen. *Monatsh. Math. Phys.*, 48:176–189, 1939.

[12] A. Poëls. On uniform polynomial approximation. *J. Éc. polytech. Math.*, 11:769–807, 2024.

[13] D. Roy. On Schmidt and Summerer parametric geometry of numbers. *Ann. of Math.*, 182:739–786, 2015.

[14] W. M. Schmidt. *Diophantine Approximation*, volume 785 of *Lecture Notes in Math.* Springer-Verlag, 1980.

[15] W. M. Schmidt and L. Summerer. Parametric geometry of numbers and applications. *Acta Arith.*, 140:67–91, 2009.

[16] W. M. Schmidt and L. Summerer. Diophantine approximation and parametric geometry of numbers. *Monatsh. Math.*, 169:51–104, 2013.

[17] V. G. Sprindzuk. Mahler's problem in the metric theory of numbers. *Amer. Math. Soc., Providence, R.I.*, 25, 1969.

[18] K. Tishchenko. On approximation to real numbers by algebraic numbers. *Acta Arith.*, 94(1):1–24, 2000.

[19] K. Tsishchanka. On approximation of real numbers by algebraic numbers of bounded degree. *J. Number Theory*, 123(2):290–314, 2007.

[20] E. Wirsing. Approximation mit algebraischen Zahlen beschränkten Grades. *J. reine angew. Math.*, 206:67–77, 1961.

Universite Claude Bernard Lyon 1,
Institut Camille Jordan UMR 5208,
69622 Villeurbanne,
France

E-mail address: poels@math.univ-lyon1.fr